Cosmology on the Ball

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1. Outline
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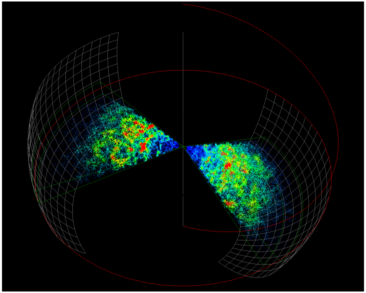
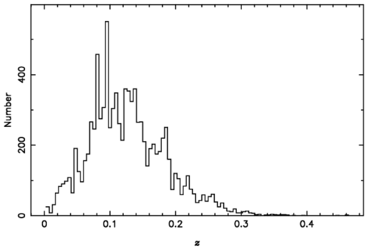
   - 3D Fourier analysis
   - 3D Spherical Fourier-Bessel analysis

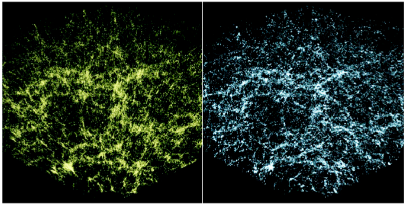
Goals



- 3D distribution $\rightarrow$ statistical information
- Configuration space $\longleftrightarrow$ harmonic space
- *n*-point correlation functions $\longleftrightarrow$ Power spectrum and *n*-spectra
- $\rightarrow$ Model parameters

Complications



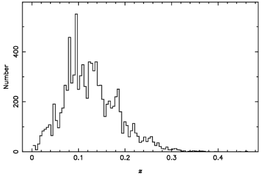


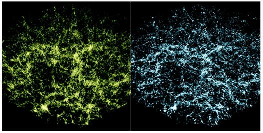
What you see is not what is there

- selection function
- distortions
- curved space

Coordinates

- Observations: right ascension α, declination δ, redshift z
- Map onto spherical coordinates $\theta = \pi - \delta,\ \varphi = \alpha$
- $r = cz/H_0$ or $c \int_0^z dz'/H(z')$
- $\Rightarrow$ spherical polars (r, θ, φ)





With some simplifications:

- Sky Mask: $n(\mathbf{r}) \rightarrow\ M(\theta, \varphi)\, n(\mathbf{r})$; $M = 0$ or 1
- Selection function: $n(\mathbf{r}) \rightarrow\ \phi(r)\, n(\mathbf{r})$; $\phi(r) \leq 1$
- Redshift-space distortions (RSD) are radial: Actually measure $\mathbf{s} = \mathbf{r} + \frac{U(\mathbf{r})}{H_0}\hat{\mathbf{r}}; \quad U(\mathbf{r}) \equiv \mathbf{v}\cdot\hat{\mathbf{r}}; \quad \mathbf{v} =$ peculiar velocity
- Time-dependent effects (growth rate etc) depend on r only
- Geometry: r may be wrong if the cosmology is wrong (via $H(z)$): Alcock-Paczynski effect

3D Fourier analysis (Kaiser 1986, Zaroubi & Hoffman 1996)

- $n^s(\mathbf{s})d^3\mathbf{s} = n(\mathbf{r})d^3\mathbf{r} \qquad n = \bar{n}(r)(1+\delta)$
- $\Rightarrow \ \delta^s(\mathbf{s}) = \delta^r(\mathbf{r}) - \left(2 + \frac{d\ln\phi}{d\ln r}\right)\frac{U(\mathbf{r})}{r} - \frac{\partial U(\mathbf{r})}{\partial r}$ (Kaiser 1986)
- In linear theory, $\mathbf{v} = -\nabla\Phi$ and Φ satisfies Poisson's equation $\nabla^2\Phi = -f(\Omega_m)\delta^r; \qquad f(\Omega_m) \simeq \Omega_m^{0.6}$
- Can sometimes ignore the middle term (not at small r)
- Standard technique: expand in Fourier basis:
- $\delta^r(\mathbf{r}) = \int d^3\mathbf{k}\, \exp(i\mathbf{k}\cdot\mathbf{r})\, \delta^r(\mathbf{k})$
- $\frac{\partial U(\mathbf{r})}{\partial r} = \int d^3\mathbf{k}\, \exp(i\mathbf{k}\cdot\mathbf{r})\, \delta^r(\mathbf{k}) \left(\frac{\mathbf{k}\cdot\mathbf{r}}{kr}\right)^2$
- Only works as a FT in the distant-observer approximation, otherwise Fourier coefficients 'depend on **r**'.

Linear theory Distortions

With a (highly-sub-optimal) weighting of the data $w(r) = 1/\phi(r)$,

- $\delta^s(\mathbf{k}) = \delta^r(\mathbf{k}) + f(\Omega_m) \int d^3\mathbf{k}' \delta^r(\mathbf{k}') \mathcal{I}(\mathbf{k}, \mathbf{k}')$
- where $\mathcal{I}$ is an integral over the survey volume V:
  $\mathcal{I}(\mathbf{k}, \mathbf{k}') = \int_V d^3\mathbf{r} \exp[i\mathbf{r} \cdot (\mathbf{k}' - \mathbf{k})] \left(\frac{\mathbf{k}' \cdot \mathbf{r}}{k' r}\right)^2$
- Elegant, simple, but it's very inconvenient...

Linear theory Distortions

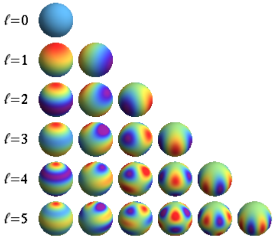
The use of exponential Fourier modes is convenient for one reason:

- Basis functions are eigenfunctions of the Laplace operator, solving
- $\left(\nabla^2 + k^2\right)\Psi = 0$
- so the peculiar velocity coefficients are simply related to the density coefficients.
- $\frac{\partial U}{\partial r}(\mathbf{k}) = \delta^r(\mathbf{k})(\hat{\mathbf{k}}\cdot\hat{\mathbf{r}})^2$.
- BUT (here with a selection function $\phi(r)$ and arbitrary weighting function $w(r)$),
- $\mathcal{I}(\mathbf{k},\mathbf{k}') = \int_V d^3\mathbf{r}\,\phi(r)w(r)\exp[i\mathbf{r}\cdot(\mathbf{k}'-\mathbf{k})]\left(\frac{\mathbf{k}'\cdot\mathbf{r}}{k'r}\right)^2$
- is a fully 6-dimensional object:
- $\delta^s(\mathbf{k}) = \int d\mathbf{k}'\,\mathcal{I}'(\mathbf{k},\mathbf{k}')\,\delta^r(\mathbf{k}')$

3D Spherical Fourier-Bessel analysis

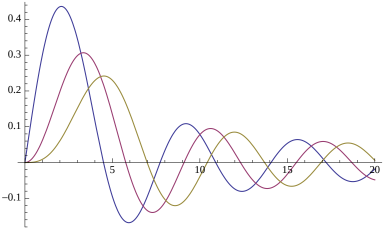
Clearly there is a more natural coordinate system. How do we benefit?

- Linear RSDs derive from a potential, so a sensible basis is the set of eigenfunctions of ∇^2
- Writing ∇^2 in spherical polars gives basis functions $j_\ell(kr)Y^*_{\ell m}(\theta,\varphi)$ so choose
- $\delta_{\ell m}(k) = \sqrt{\frac{2}{\pi}}\int d^3\mathbf{r}\,\delta(\mathbf{r})\,k\,j_\ell(kr)\,Y^*_{\ell m}(\theta,\varphi)$
- $\langle\delta_{\ell m}(k)\delta^*_{\ell' m'}(k')\rangle = P(k)\delta^D(k-k')\delta^K_{\ell\ell'}\delta^K_{mm'}$



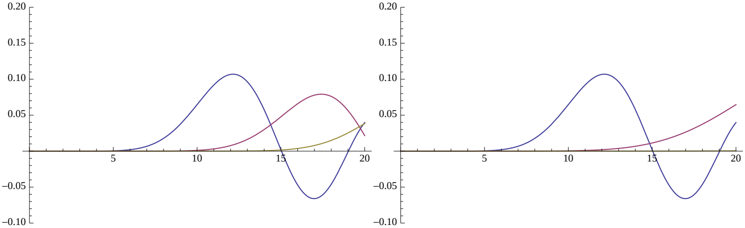
Spherical polars (Heavens & Taylor 1995)

- Boundary conditions: either specify on a sphere of (large) radius, or let radius $\to \infty$
- Gives discrete $(k_{\ell n}; n = 0, 1, \ldots)$, or continuous, set of k values
- Boundary condition: several considered. e.g. zero velocity, which avoids RSD boundary terms, but which requires $\langle \text{field} \rangle = 0$
- Can generalise to curved space, by using hyperspherical Bessel functions



Spherical polars (Heavens & Taylor 1995)

- Continuous k: values need to be chosen carefully to avoid strong covariance between modes. Leads to similar modes to $k_{\ell n}$
- Very low k, or very high ℓ values are $\sim$zero within the survey volume $\Rightarrow$ limits the range of k. (First peak is at $r \sim \ell/k$)



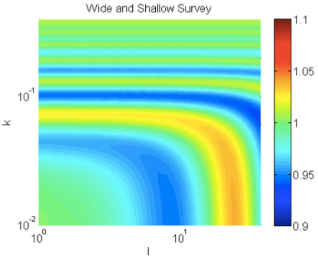
Linear theory Distortions in spherical coordinates

Expanding in the eigenfunctions of ∇^2, but in spherical coordinates, in redshift-space, there are analogous relations to the Fourier analysis. For all-sky, ℓ and n are coupled:

- $\delta^s_{\ell mn} = \left(\phi^{\ell' n'}_{\ell n} + \frac{f(\Omega_m)}{b} V^{\ell' n'}_{\ell n}\right) \delta^r_{\ell' mn'}$
- With a mask, an angular convolution W is added, coupling ℓ and m:
- $\delta^s_{\ell mn} = W^{\ell'' m'}_{\ell m} \left(\phi^{\ell' n'}_{\ell'' n} + \frac{f(\Omega_m)}{b} V^{\ell' n'}_{\ell'' n}\right) \delta^r_{\ell' m' n'}$
- $W^{\ell' m'}_{\ell m}, \phi^{\ell' n'}_{\ell n}$ and $V^{\ell' n'}_{\ell n}$ are one-off integrals over the angular mask and radial selection function
- They clearly separate the effects of mask, selection function and distortion
- They are 4-dimensional objects, and make the analysis tractable

Generalisations and applications

- PSCz/2dF analysis (Tadros et al 1999, Percival et al 2004)
- Works for wide-angle separations when Kaiser analysis fails (e.g. Raccanelli et al 2010)
- Baryon Acoustic Oscillations (BAOs; Rassat & Refregier 2012, Pratten & Munshi 2013)
- `3DEX pycosmo` code (Leistedt et al. 2012)
- High ℓ, low k (with dropping selection function): cosmology on the ball $\rightarrow$ cosmology on the sphere



Generalisations and applications (continued)

- Line-of-sight CMB analysis (Abramo et al 2010)
- Gravitational lensing (Heavens 2003… Kitching et al 2014). Here we take $\gamma_{\ell m}(k) = \int d^3\mathbf{r}\, j_\ell(kr)\, {}_2Y^*_{\ell m}(\theta, \varphi)$ since the shear field is a spin-weight 2 field. Dependence on a physical wavenumber k allows one to isolate physical scales (e.g. avoid highly non-linear regime in lensing)

Summary

- Spherical polars are obviously the natural coordinate system to use
- Most effects are either radial or tangential
- Advantage shows in relation of observed variables to underlying variables (4D rather than 6D)
- Spherical Bessel functions + spherical harmonics are natural for fields concerned with potentials
- Other functions are available