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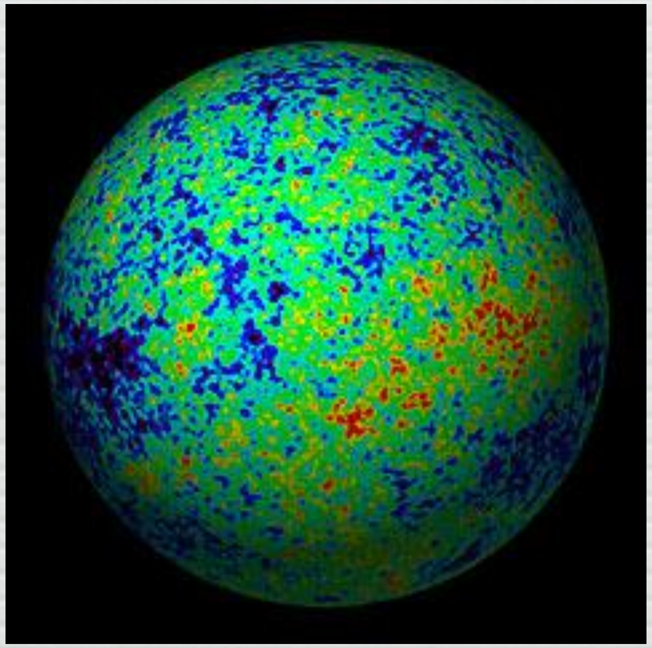
The Cosmological Context: CMB analysis formalism



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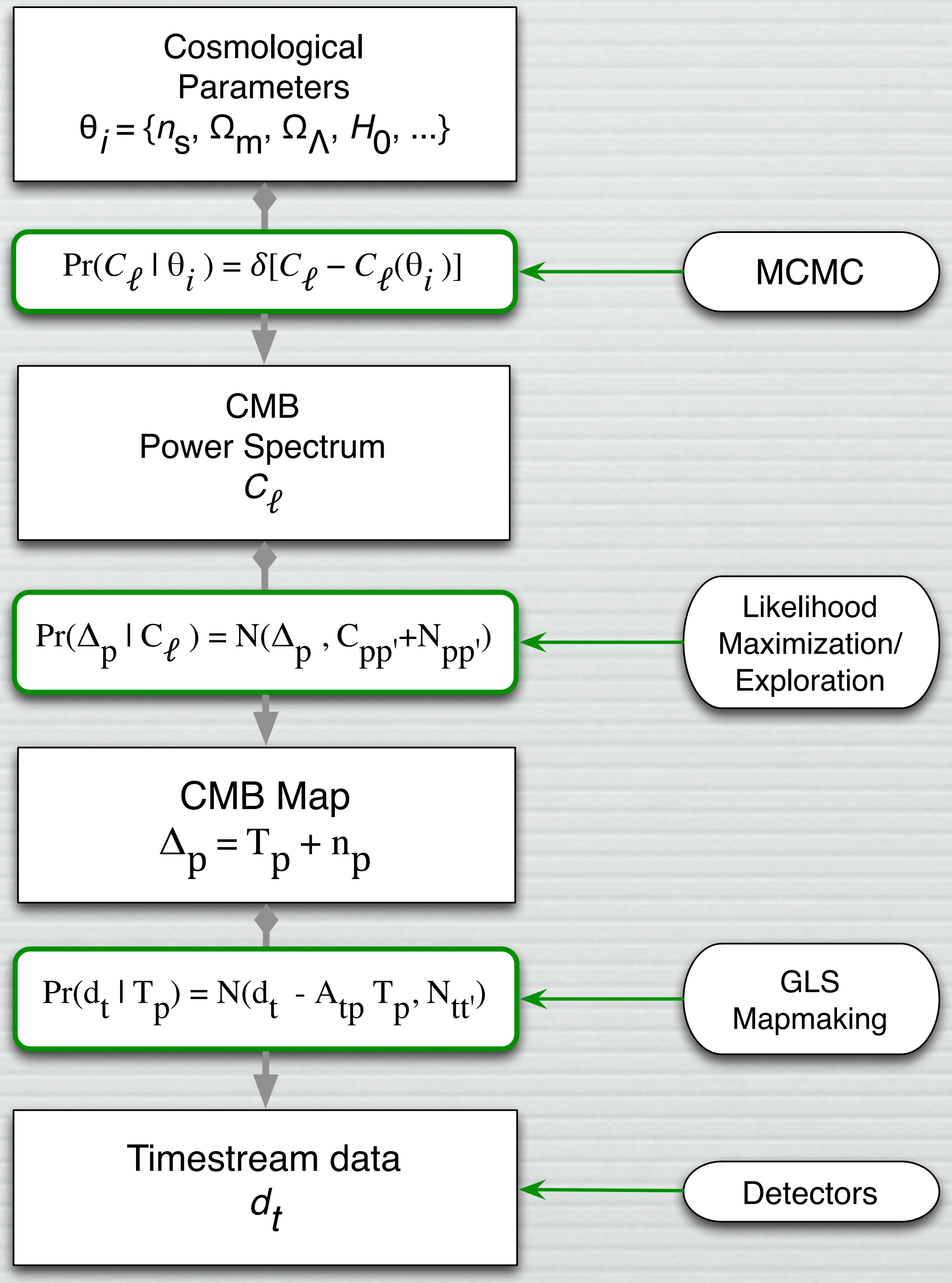
Science on the Sphere
July 2014

ICIC Imperial Centre for Inference & Cosmology

Parts of this talk may have been based on data and ~30 papers released March 2013 by the 400-person Planck Collaboration

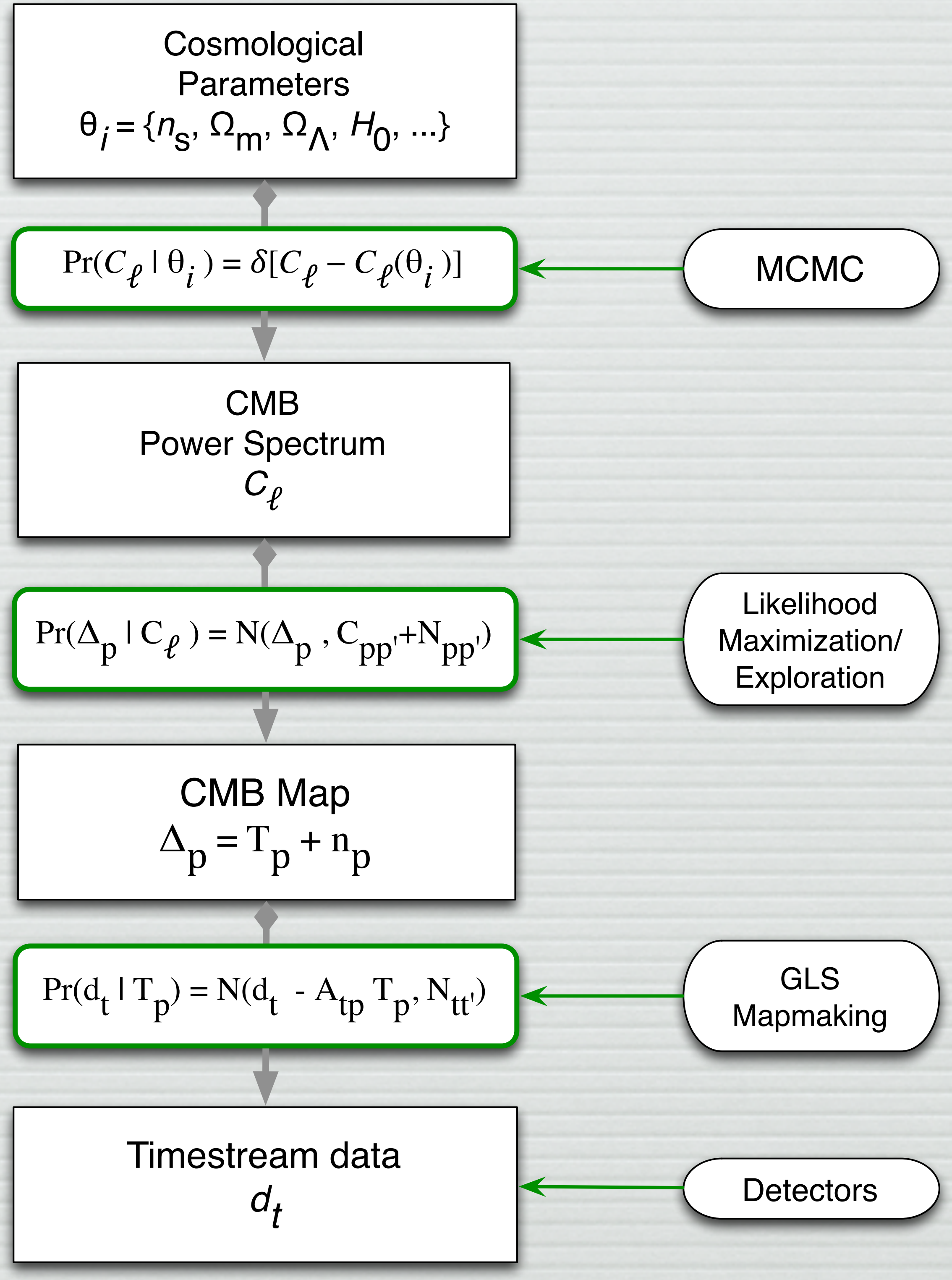
Analysing CMB data (on the sphere)

- Where does the spherical sky enter?
  - Theory
    - $C_\ell = C_\ell[\{\theta_i\}]$
  - Power spectra
    - map → $a_{\ell m}$ → C_ℓ
    - spherical harmonic transform: $Y_{\ell m}$
    - but: mask and ***noise***
  - Mapmaking
    - pixelization of the sky
  - Polarization
    - EB decomposition is specific to the sphere
      - (equivalents exist for other manifolds, including flat)



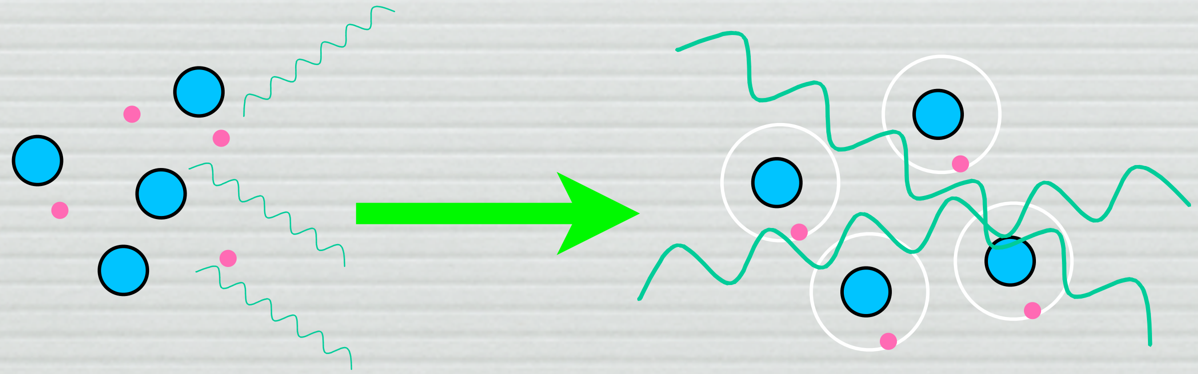
Analysing CMB data (on the sphere)

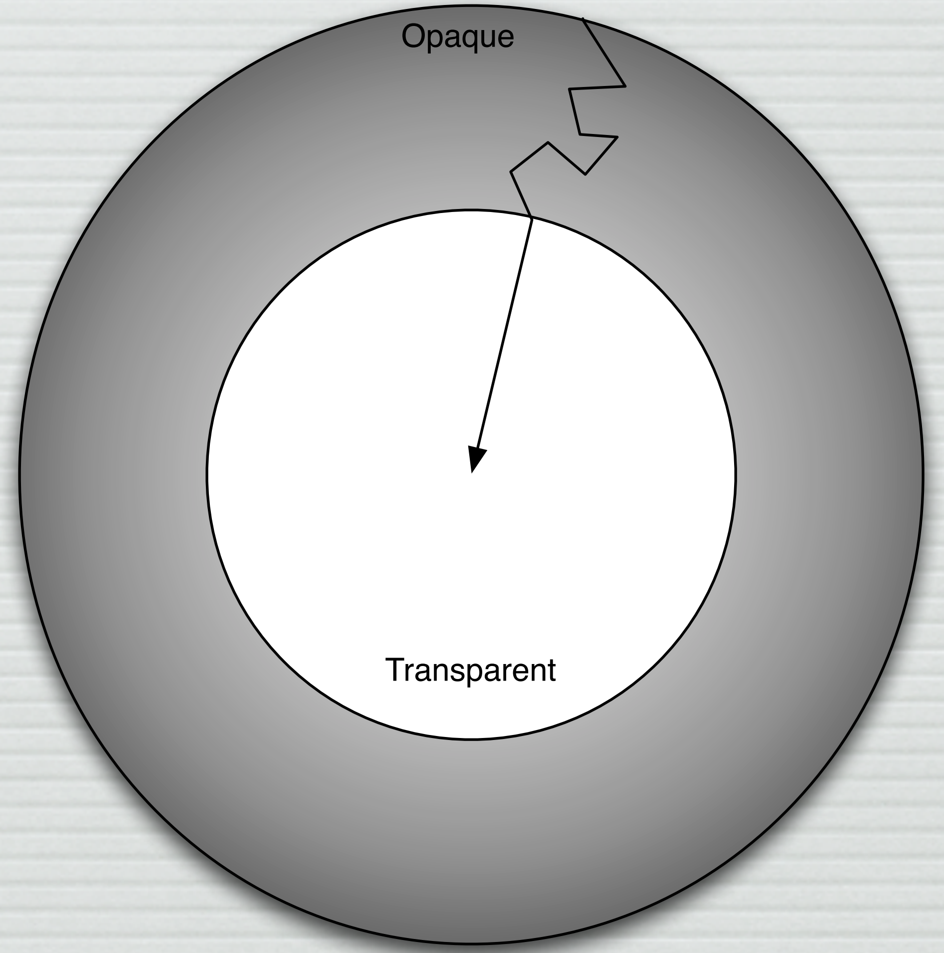
- CMB as a hierarchical model
  - can be computed exactly using Gibbs methods, approximately using approximations for $\Pr(\hat{C}_\ell | C_\ell)$
- Map and power spectrum are just (approximately) sufficient statistics
- Radical compression (~*sparsity*):
  - 10^{12} samples → 10^7 pixels → 10^3 C_ℓ → 6 parameters
- This version assumes
  - isotropic Gaussian signal (no topology)
  - known & Gaussian noise properties
  - known (isotropic) beam shape
  - no foregrounds
  - no systematics
- Even so: compute-bound $\mathcal{O}(N_{pix}^3)$:
  - covariance matrix in mapmaking
  - likelihood evaluation in C_ℓ step



Evidence & Observations: Cosmic Microwave Background

- 400,000 years after the Big Bang, the temperature of the Universe was $T \sim 3{,}000$ K
- Hot enough to keep hydrogen atoms *ionized* until this time
  - ***proton + electron → Hydrogen + photon*** $[p^+ + e^- \rightarrow H + \gamma]$
  - ***charged plasma → neutral gas***
    - depends on *entropy* of the Universe
- Photons (light) can't travel far in the presence of charged particles
  - ***Opaque → transparent***



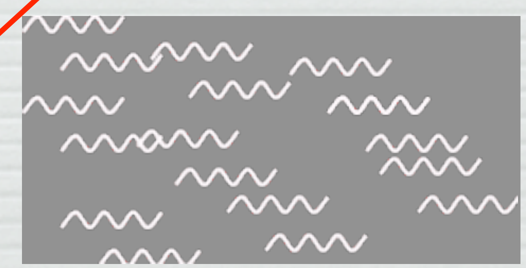


What affects the CMB temperature?

$$\frac{\Delta T}{T}(\hat{x}) \simeq \frac{1}{4}\frac{\delta\rho_\gamma}{\rho_\gamma} + v\cdot\hat{x} + \int_{\eta_{rec}}^{\eta_0} d\eta\, \dot{h}_{ij}\hat{x}_i\hat{x}_j$$

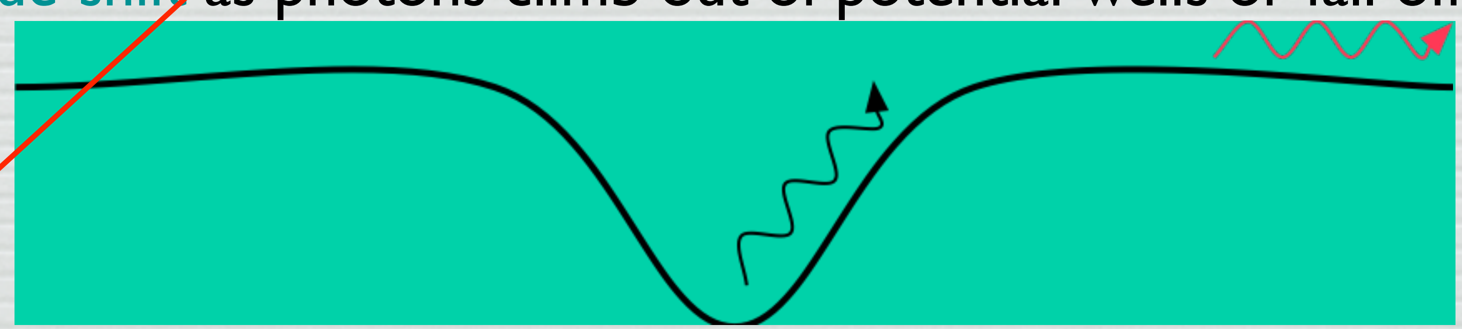
- Initial temperature (density) of the photons

Cooler



Hotter

- Doppler shift due to movement of baryon-photon plasma
- Gravitational red/blue-shift as photons climb out of potential wells or fall off of underdensities



- Photon path from LSS to today
- All linked by initial conditions ⇒ 10^{-5} fluctuations

CMB: from theory to statistics

- Start with 3D fields
  - photon distribution function
  - gravitational potential (metric)
  - density of matter components (dark matter, electrons, atoms, …)
  - all linked by physics and initial conditions
    - early Universe: small fluctuations, approximately Gaussian
    - linear evolution $\Rightarrow$ preserves (isotropic) Gaussian distribution

CMB Statistics

z~1300: p+e→H & Universe becomes transparent.

$$\frac{T(\hat{x})-\bar{T}}{\bar{T}} \equiv \frac{\Delta T}{T}(\hat{x}) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\hat{x})$$

i.e., Fourier Transform, but on a sphere

Determined by **temperature**, **velocity** and **metric** on the last scattering surface.

Power Spectrum:

$$\langle a^*_{\ell m} a_{\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'}\, C_\ell$$

Multipole ℓ ~ angular scale $180°/\ell$

For a **Gaussian** theory, C_ℓ completely determines the statistics of the temperature — and is determined by the cosmological parameters

The CMB transfer function

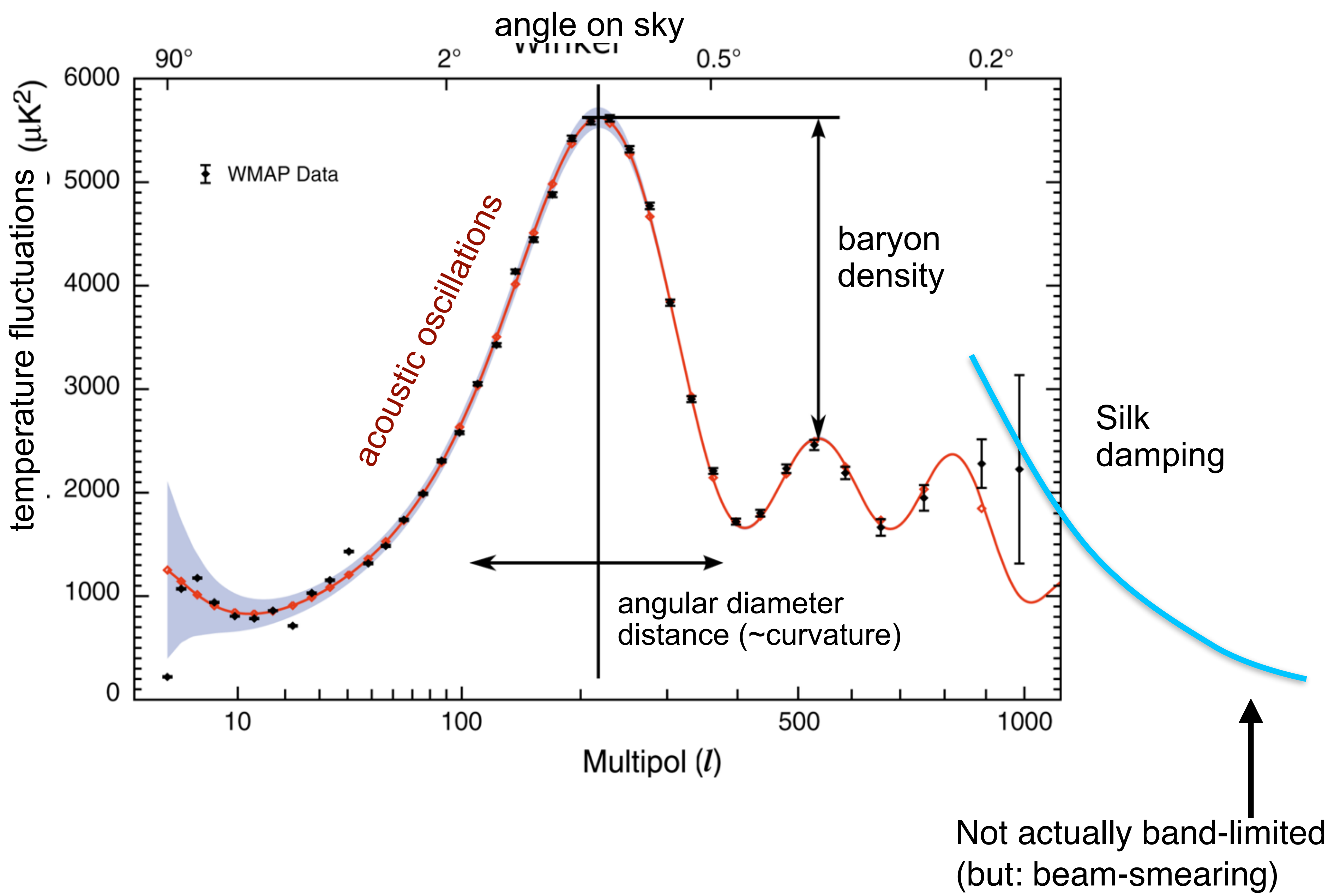
$$C_\ell = \int P_i(k) T_\ell^2(k)\; dk$$

- compare density spectrum: $P(k) = P_i(k)T^2(k)$
- The transfer function depends on the "cosmological parameters". For example:
  - matter density—determines sound speed in baryon/ photon fluid
  - curvature—determines angular-diameter distance to horizon

  - Actually solve Boltzmann Equation over thickness of Last-Scattering surface – e.g., `CMBFAST, CAMB`

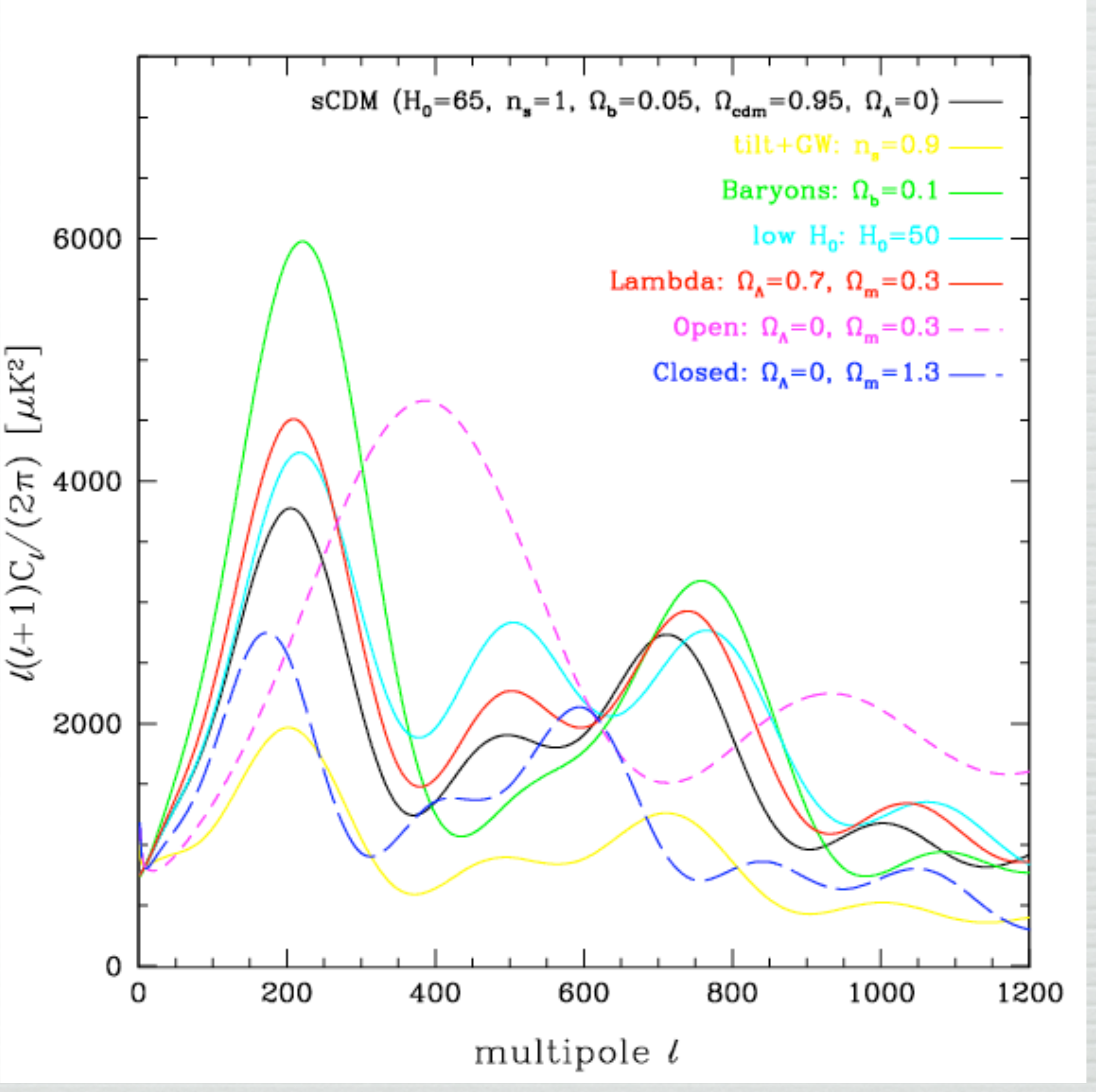
Physics of the CMB power spectrum

Gravity + plasma physics modulates initial spectrum of fluctuations (from, e.g., inflation)



Theoretical Predictions

Mean square fluctuation amplitude



~180°/Angular scale

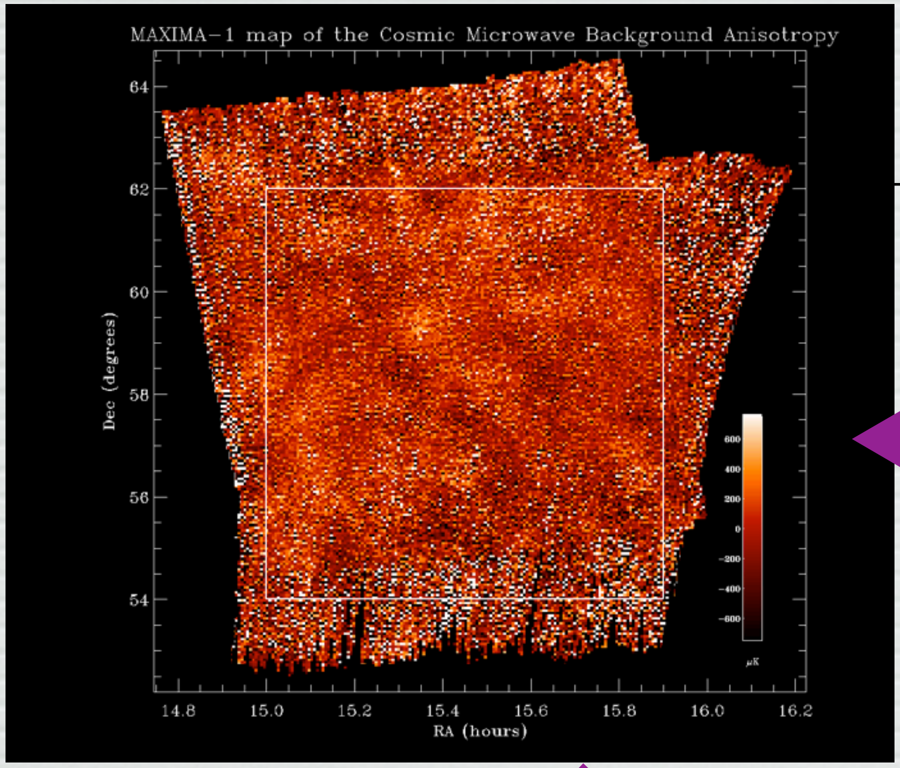
Mapmaking: Likelihood Function

- data model: $d_t = A_{tp} T_p + n_t$
  - "design matrix" A_{tp} contains pointing information
  - $\langle n_t n_{t'} \rangle = N_{tt'} = N(t-t')$ [Fourier Tr. of $N(f)$]
  - stationary, Gaussian noise:
    - $$P(d|TI) = \frac{1}{|2\pi N|^{1/2}} \exp\left[-\frac{1}{2}(d-AT)^T N^{-1}(d-AT)\right]$$
    - this is a "generalized linear model"

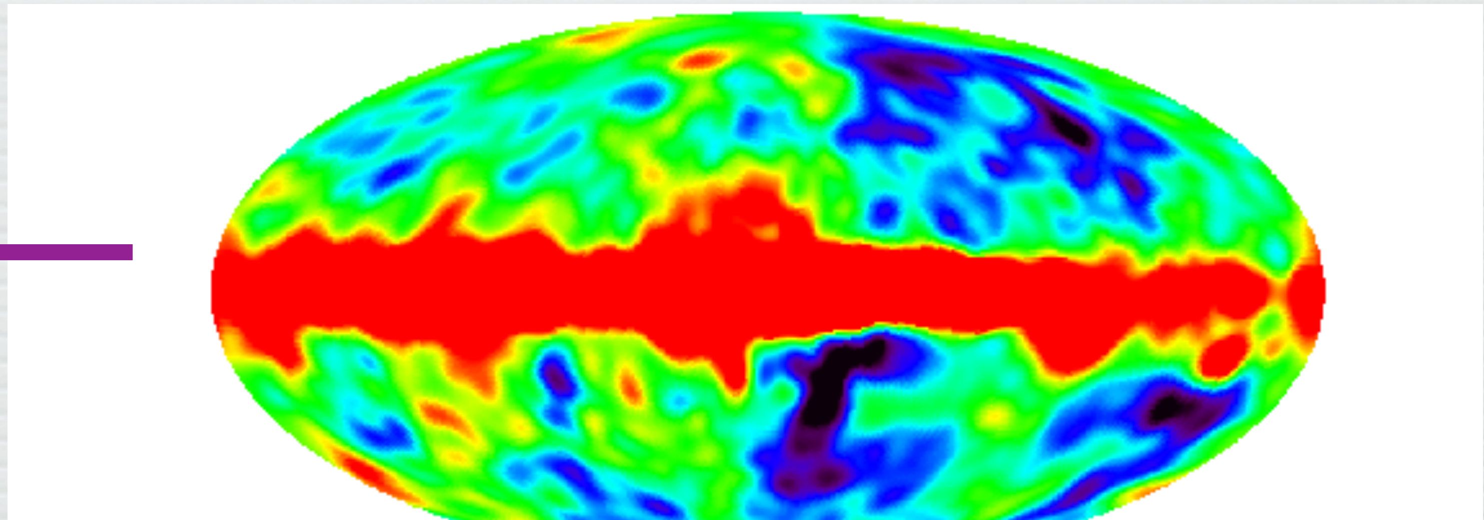
$$\bar{T}_p = (A^T N^{-1} A)^{-1} A^T N^{-1} d \qquad \langle \delta T_p \delta T_{p'} \rangle = (A^T N^{-1} A)^{-1}$$

- Sphere only arises through locations of "output" pixels T_P

Maps of the Cosmos

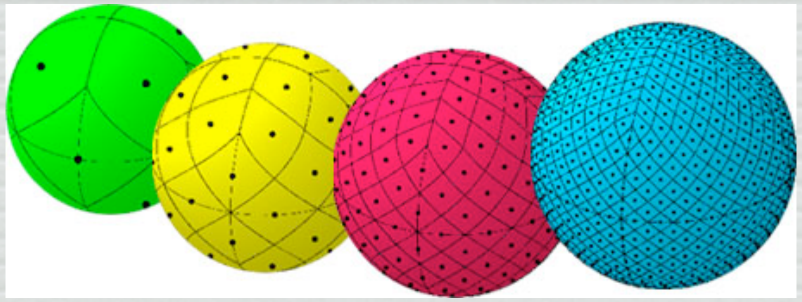


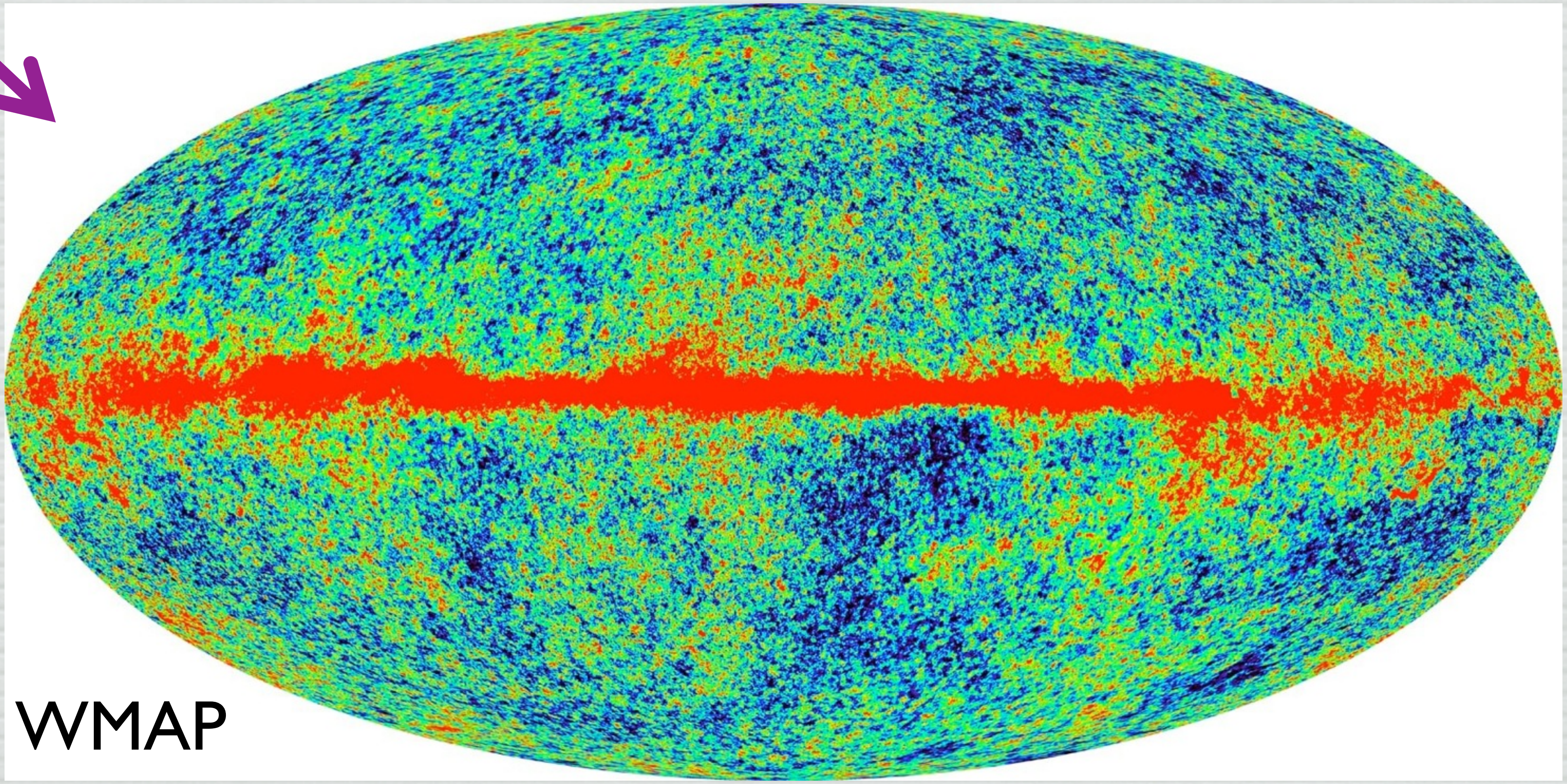
MAXIMA



COBE/DMR

NB. pixelization on sphere non-trivial. CMB uses "HEALPix"





WMAP

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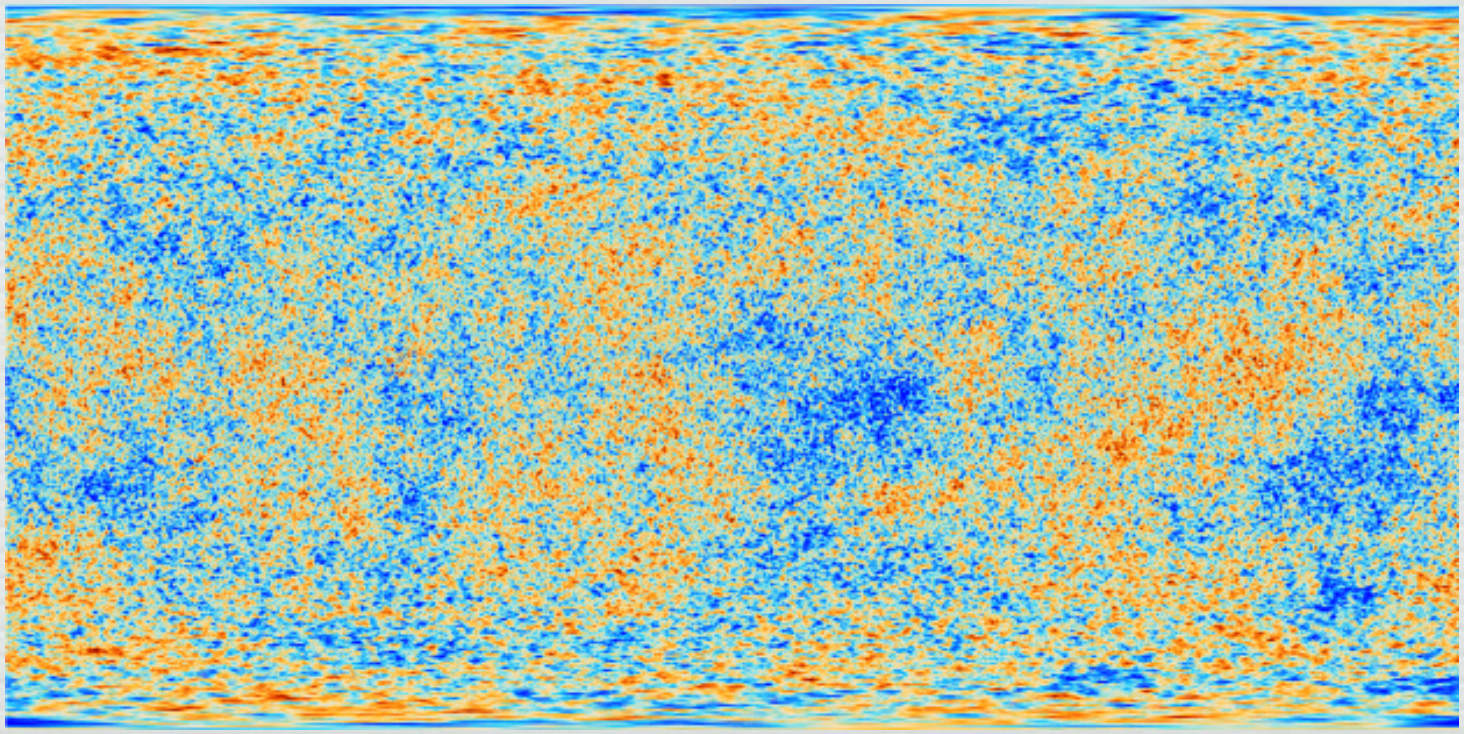
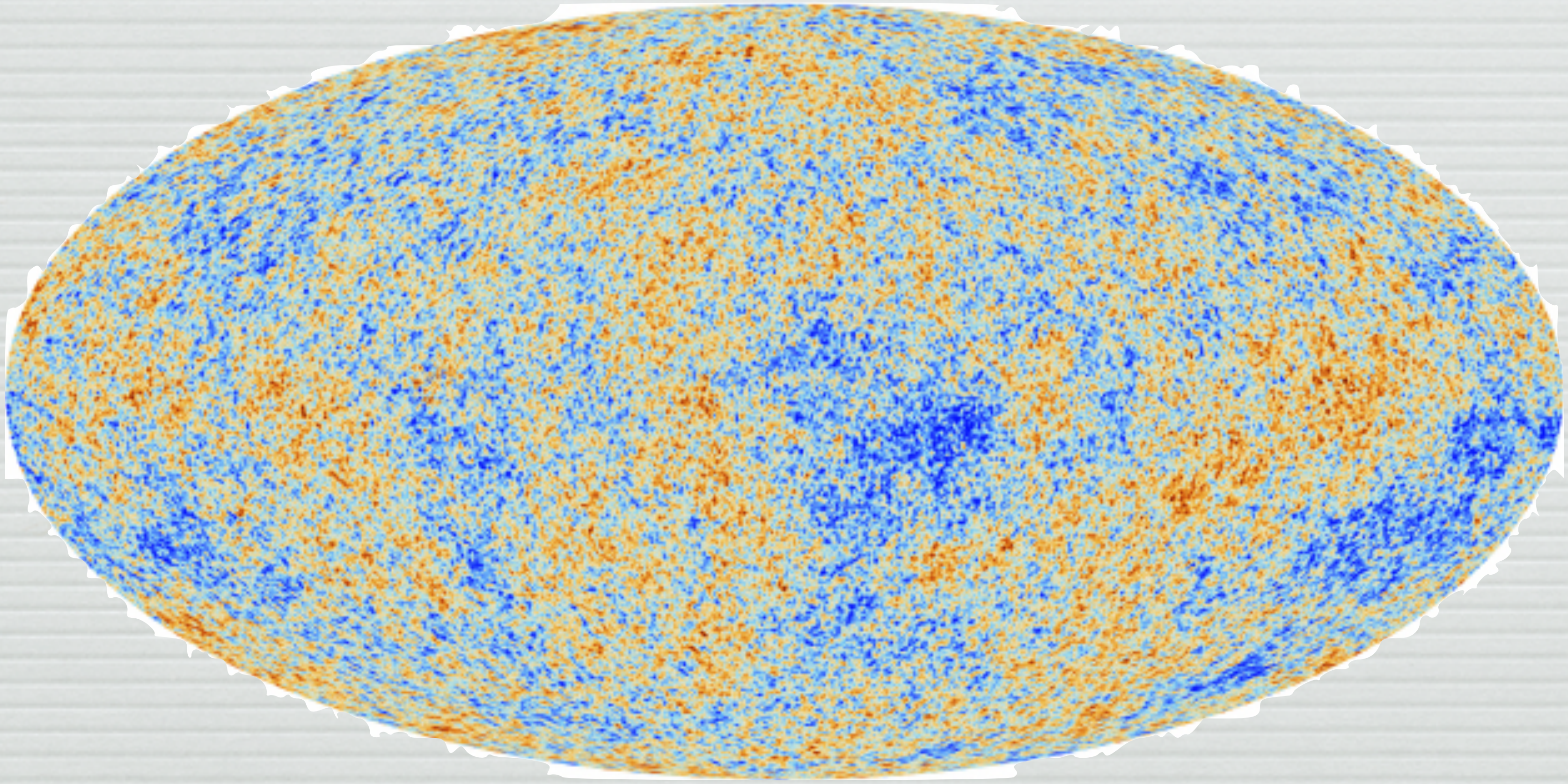
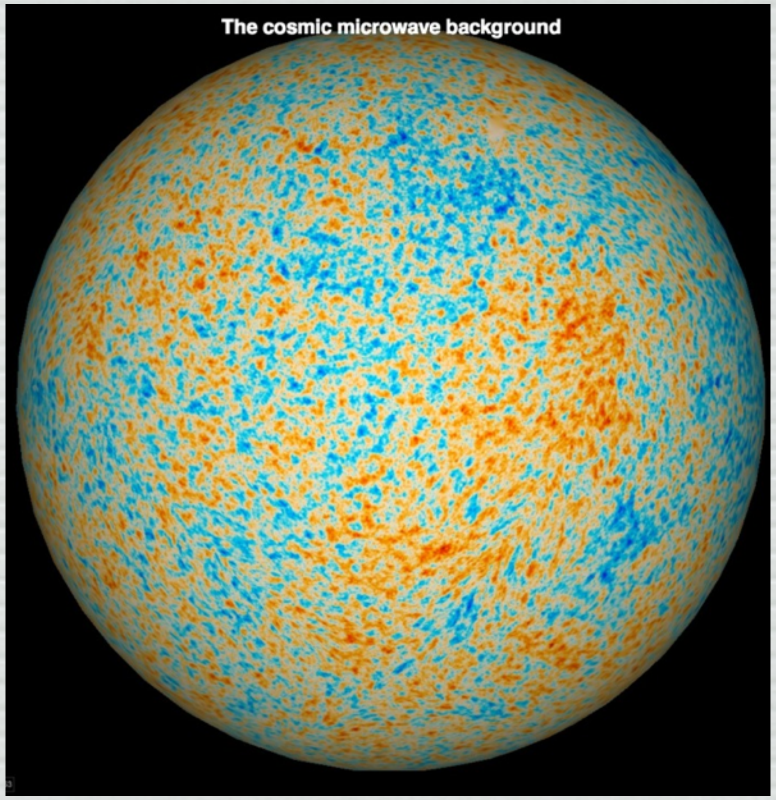
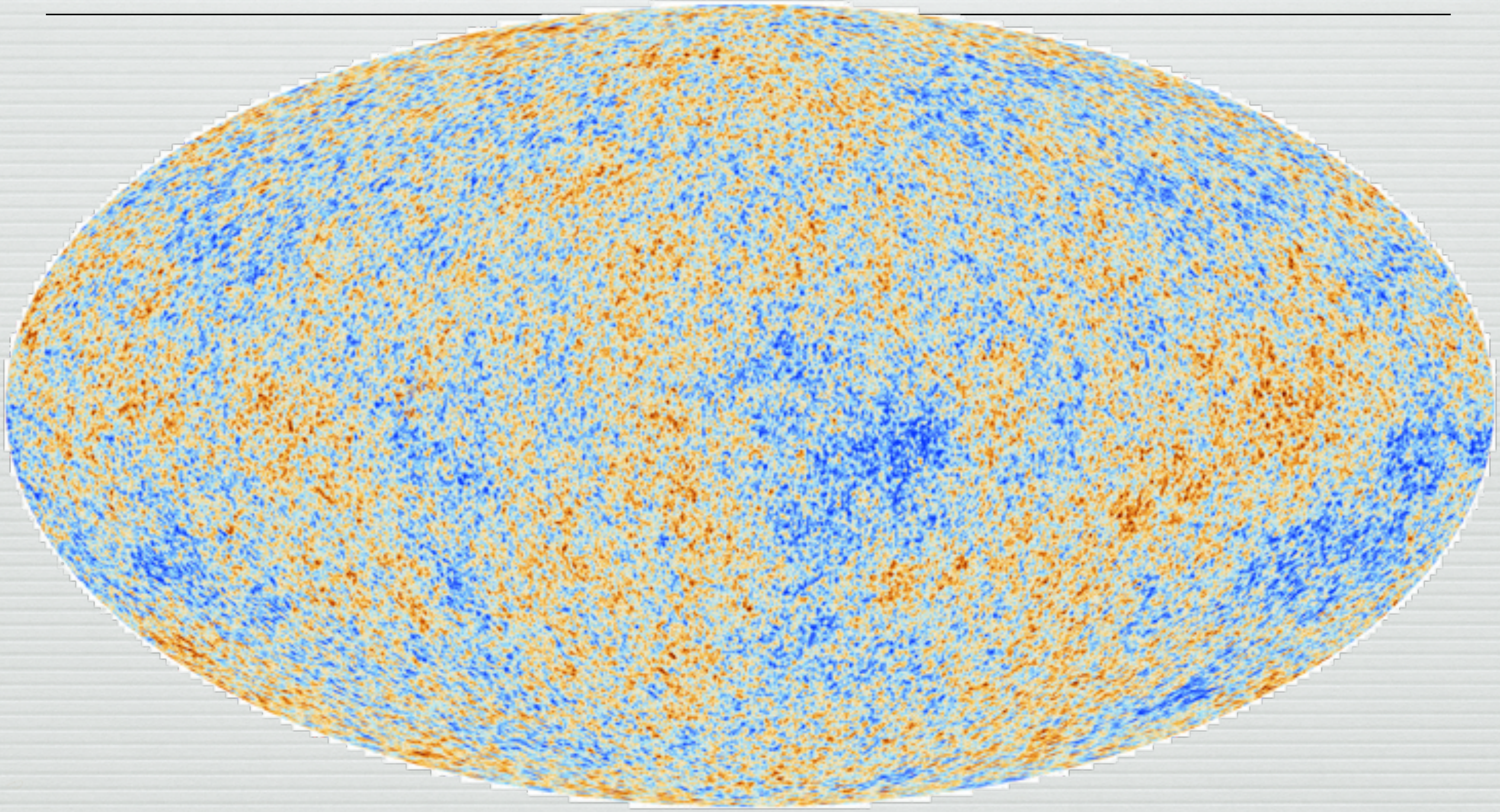
Pixels and projections







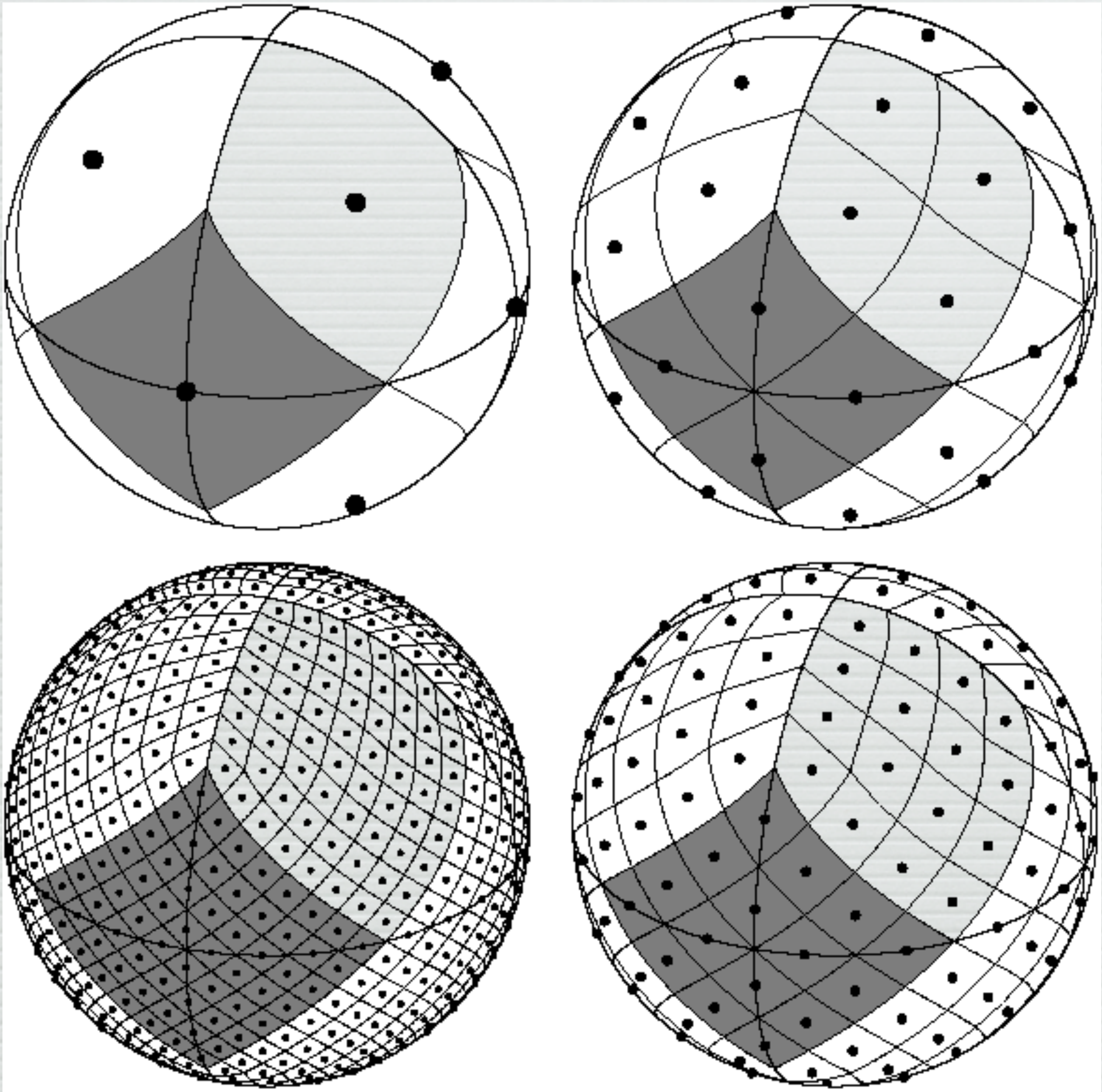
Planck

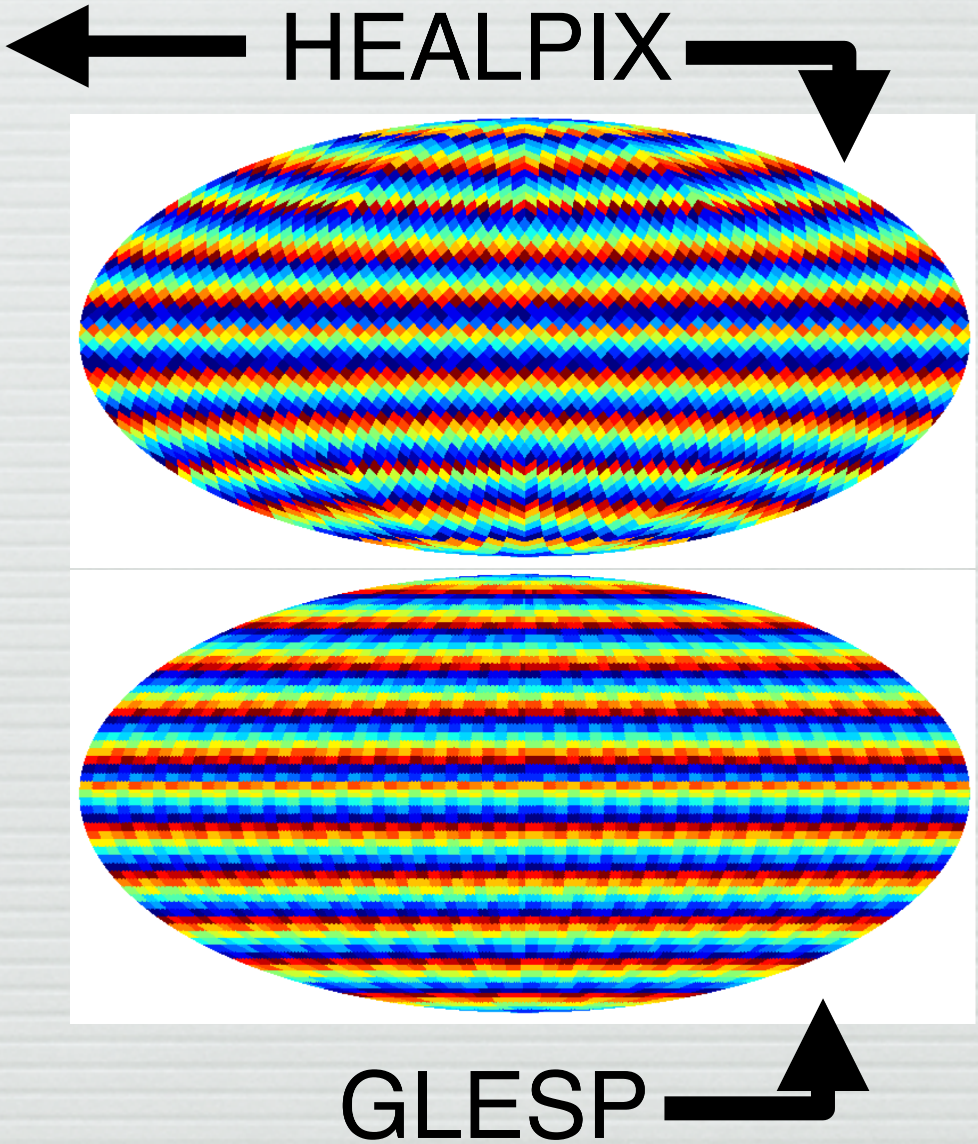


Pixelization

- Competing desiderata
  - fast, approximate, transforms
    - latitudinal pixels
  - "sampling-friendly"
    - i.e., minimal # to reproduce harmonic content to a given ℓ
  - compact, equal-area pixels
    - so pixel smoothing ~isotropic and ~constant across map
  - hierarchical
    - easy to switch between pixel sizes
  - *nb. need to define full pixels, not just points*
    - *we don't sample, we convolve*
- Current favourite: HEALPIX (Gorski et al 2005)
  - Hierarchical Equal Area isoLatitude Pixelization

Pixelization





- Gauss-Legendre Sky Pixelization
  - exact quadrature in $x=\cos\theta$
  - non-equal-area pixels
  - Doroshkevich et al 2003,05,09

CMB Data Analysis: Spectrum estimation

- Model the sky as a correlated, statistically isotropic Gaussian random field

$$\frac{T(\hat{x}) - \bar{T}}{\bar{T}} \equiv \frac{\Delta T}{T}(\hat{x}) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\hat{x}) \qquad \langle a^*_{\ell m} a_{\ell' m'} \rangle = \delta_{\ell\ell'}\delta_{mm'} C_\ell$$

Parametric version of cov. mat. est'n: diag in ℓ basis

$$\langle T_p T_{p'} \rangle = S_{pp'} = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell B_\ell^2 P_\ell(\hat{x}_p \cdot \hat{x}_{p'})$$

spherical harmonic wavenumber ℓ

$$P(\bar{T}|C_\ell) = \frac{1}{|2\pi(S+N)|} \exp -\frac{1}{2}\bar{T}^T (S+N)^{-1}\bar{T}$$

- goal: characterise this over the space of C_ℓ (or parameters in $C_\ell = C_\ell[\{\theta_i\}]$
- direct computation prohibitive for high-res/full-sky
    - complicated and expensive function of C_ℓ
        - Many practical issues in calculating this explicitly.
        - At low ℓ, use sampling (usu. Gibbs), Newton-Raphson, Copula
        - At high ℓ, approximate by a function of estimated (ML) C_ℓ and errors & some other information X_ℓ

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Pseudo-spectra

- In practice, this Bayesian method is too expensive.
- What is C_ℓ?
  - Variance of distribution?
  - Sky average? $\frac{1}{2\ell+1}\sum_\ell |a_{\ell m}|^2$
  - Ergodic average? $\left\langle |a_{\ell m}|^2 \right\rangle$

} Coincide for full-sky, Gaussian distributed, noise-free map

Use "pseudo-C_ℓ": $\widehat{D}_\ell = \frac{1}{m}\sum_m |d_{\ell m}|^2 \qquad d_{\ell m} = \sum_p d(\hat{x}_p) Y_{\ell m}(\hat{x}_p)$

- this combination only appears in likelihood function in the limit of full sky and uniform noise
- w/ some sort of apodization/weighting
- frequentist linear reweighting/debiasing

$$\left\langle M_{\ell\ell'} \widehat{D}_\ell + N_\ell \right\rangle = C_\ell$$

The (more realistic) CMB

- Cannot (computationally) afford idealogical purity
  - $Y_{\ell m}$ are no longer good eigenmodes in the presence of noise and mask
    - KL (aka SN-eigenmode) is general solution, but bespoke (and very expensive)
- Use (MASTER-like) frequentist algorithms and interpret results as if they are Bayesian
  - i.e., assume asymptotic limit, and analogies with full-sky uniform-noise case which has numerical agreement between MLE and frequentist mean (sufficient statistics)
- Component separation
  - *priors* motivate inference and algorithms
    - physical information about components
    - mathematical classification of foreground-signal properties — the sphere
      - e.g., use of (various species of) spherical wavelets

CMB Likelihoods: cosmology and foregrounds

- Model residual foreground contamination as an isotropic [~Gaussian] field with known (or parameterised) spectral shape

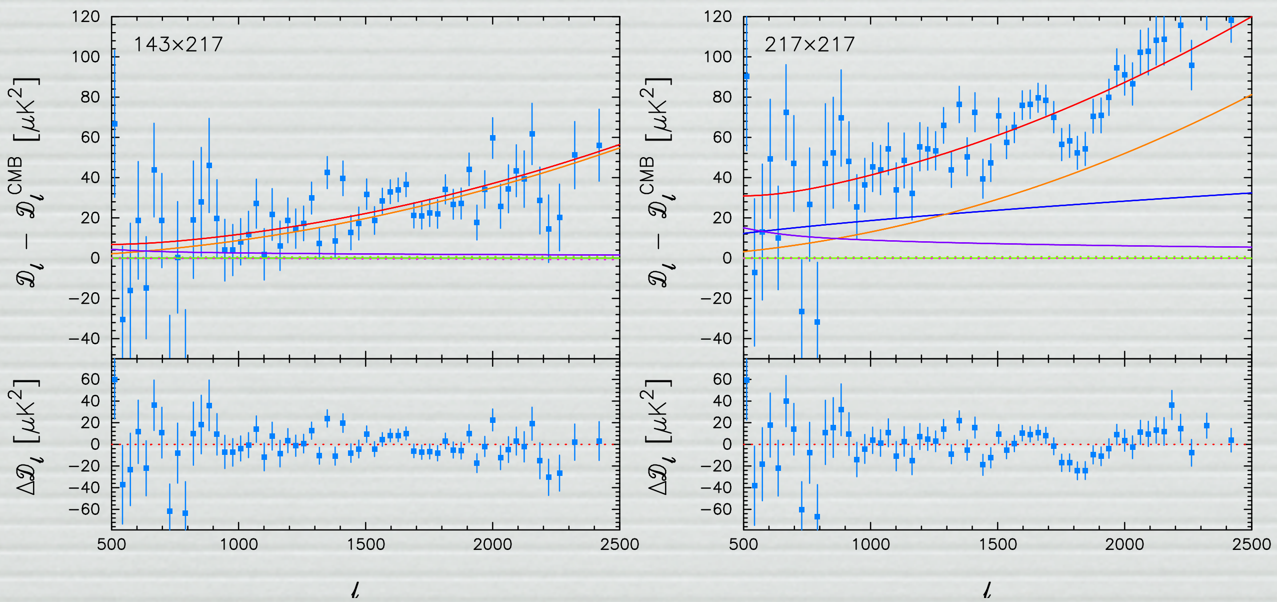


Figure 13. Foreground model over the full range of HFI cosmological frequency combinations. The upper panel in each plot shows the residual between the measured power spectrum and the 'best-fit' primary CMB power spectrum, i.e., the unresolved foreground residual for each frequency combination. The lower panels show the residuals after removing the best-fit foreground model. The lines in the upper panels show the various foreground components. Major foreground components are shown by the solid lines, colour coded as follows: total foreground spectrum (red); Poisson point sources (orange); CIB (blue); thermal SZ (green). Minor foreground components are shown by the dotted lines: kinetic SZ (green); tSZ X CIB cross correlation (purple). The 100×143 and 100×217 GHz spectra are not used in the CamSpec likelihood. Here we have assumed $r^{PS}_{100\times143} = 1$ and $r^{PS}_{100\times217} = 1$.

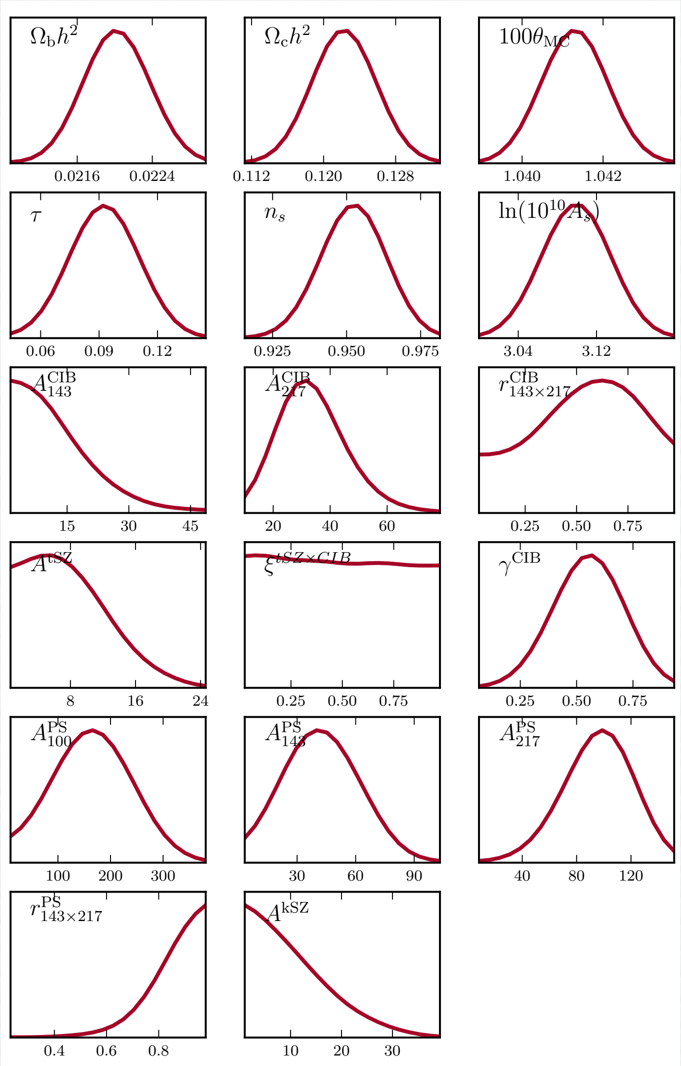


Figure 12. Marginal posterior distributions for the six cosmological (top two rows) and eleven nuisance parameters (lower four rows) estimated with the CamSpec likelihood.

Spectra and likelihoods

- Cosmologists don't (shouldn't) actually care about C_ℓ.
  - What we really want is a good way to compress the likelihood $P(\mathrm{cosmology} \mid \mathrm{data})$
  - $P(d|\theta) = P(d|C_\ell[\theta])$
    - In general, complicated O(N³) fn of the data, but there are various ansatze

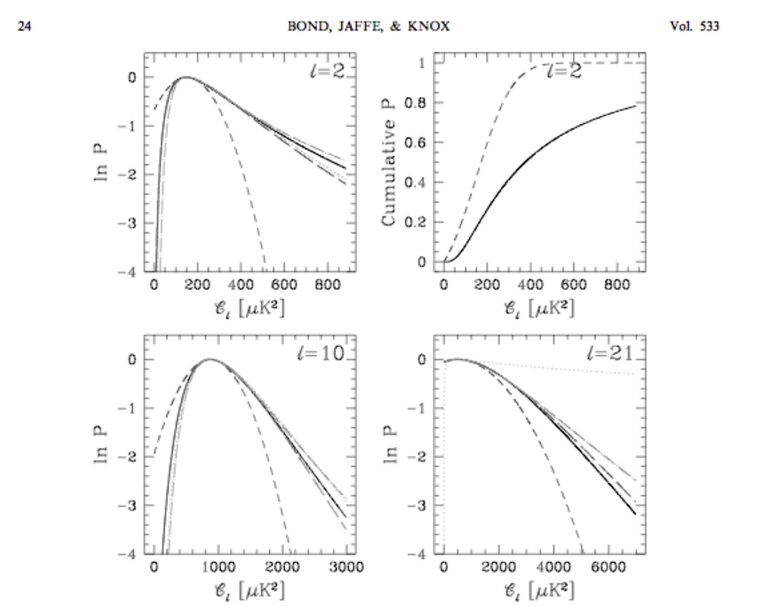
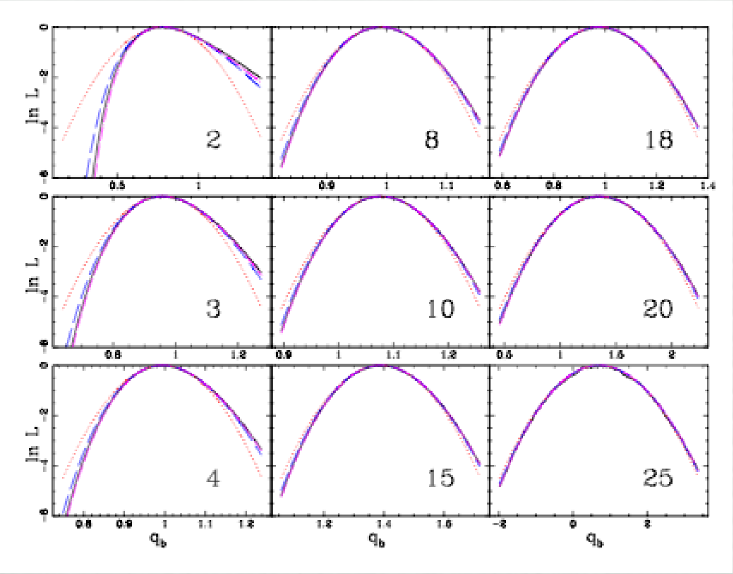


Fig. 2.—Full and approximate COBE/DMR likelihoods $P(\Delta|\mathcal{C}_\ell)$ for various values of ℓ, as marked. The horizontal axis is $\mathcal{C}_\ell = \ell(\ell+1)C_\ell/(2\pi)$. The upper right panel gives the cumulative probability. The solidline is the full likelihood calculated exactly. The short-dashed line is the Gaussian approximation about the peak. The dottedline is a Gaussian in $\ln \mathcal{C}_\ell$; the dashedline is a Gaussian in $\ln(\mathcal{C}_\ell + x_\ell)$, as discussed in the text. The dot-dashed line is the equal-variance approximation.



Expected errors

- Estimating the error (variance$^{1/2}$) on a variance (C_ℓ)
- $\langle \delta C_\ell \, \delta C_\ell \rangle = \langle a_{\ell m} \, a_{\ell m} \, a_{\ell m} \, a_{\ell m} \rangle - \langle a_{\ell m} \, a_{\ell m} \rangle \langle a_{\ell m} \, a_{\ell m} \rangle$
  - Wick's theorem: $\langle a^4 \rangle = 3 \langle a^2 \rangle^2$
  - CMB case: Knox 95, Hobson & Magueijo 96
    - need to account for $(2\ell+1) f_{\rm sky}$ measurements of each ℓ

$$(\delta C_\ell)^2 \cong \frac{2}{(2\ell+1) f_{\rm sky}} (C_\ell + N_\ell)^2 \qquad N_\ell \approx w^{-1} = (\theta_p \sigma_p)^{-2}$$

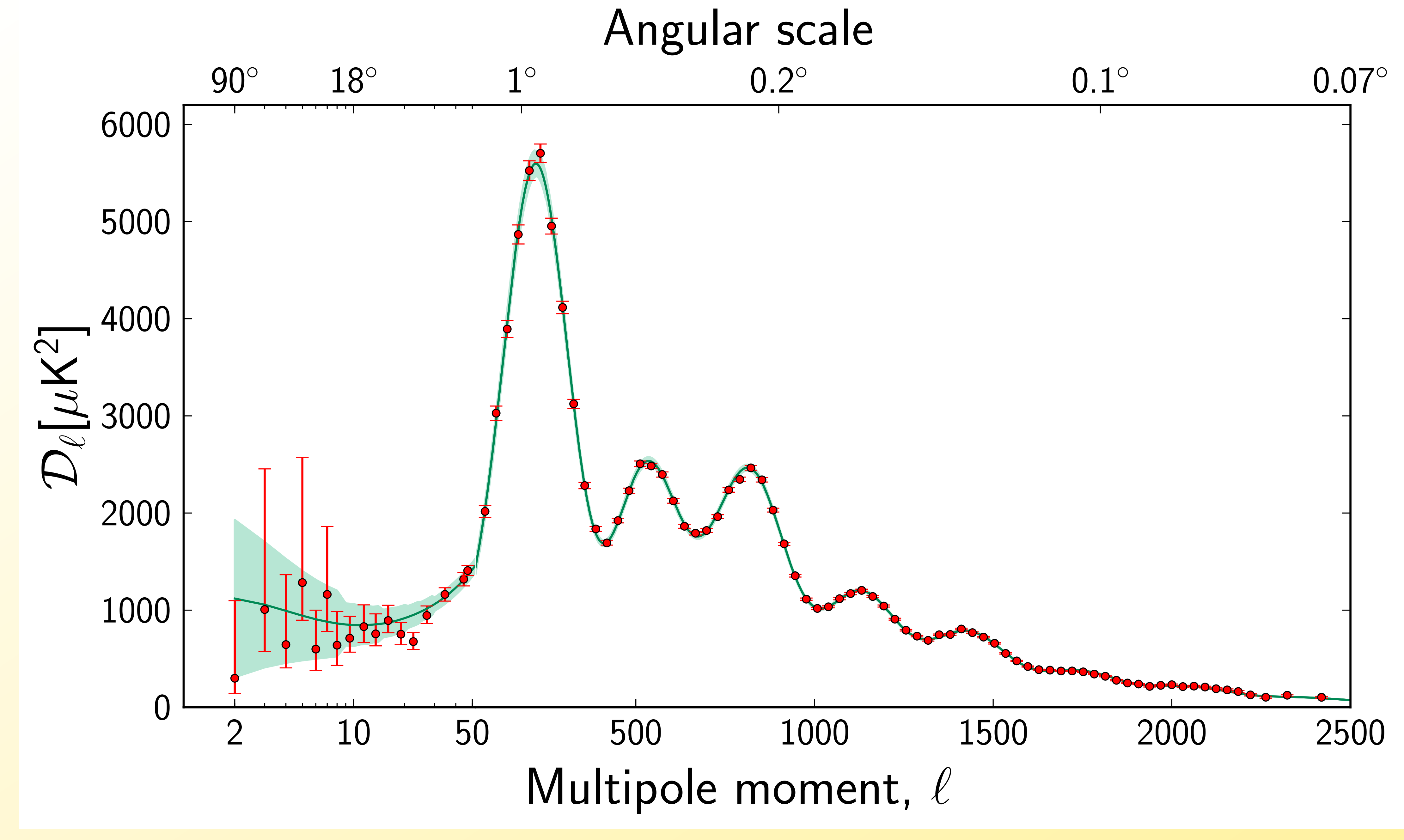
# of modes

Sample (cosmic) Variance

Noise variance ~(weight per sr)

    - Bandpowers: bin in ℓ (weighted for specific C_ℓ shape) to reduce errors and decrease covariance

Planck errors



Error band: cosmic variance estimate
error bars: cosmic + noise variance

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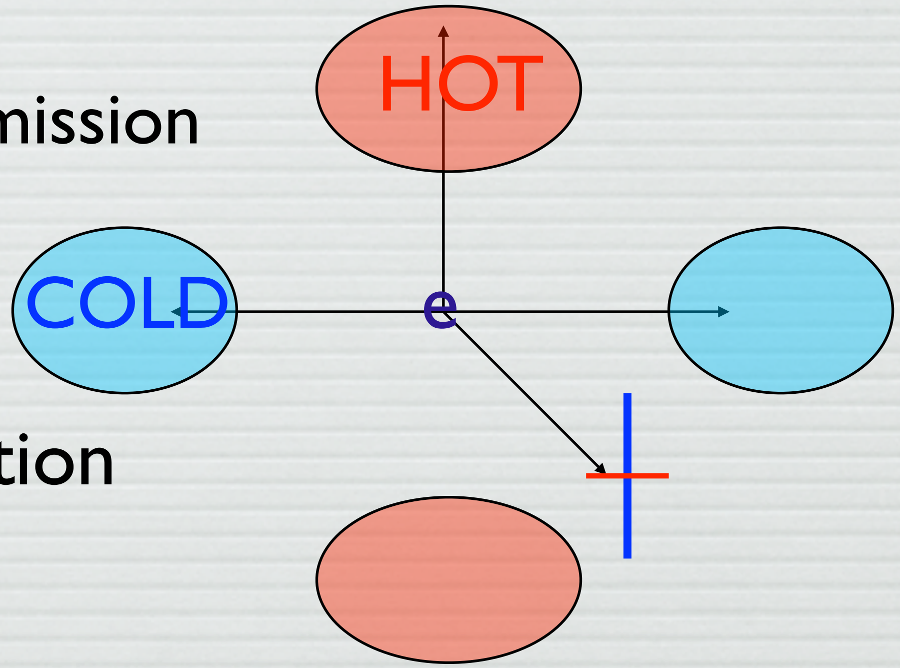
Polarization: Physics

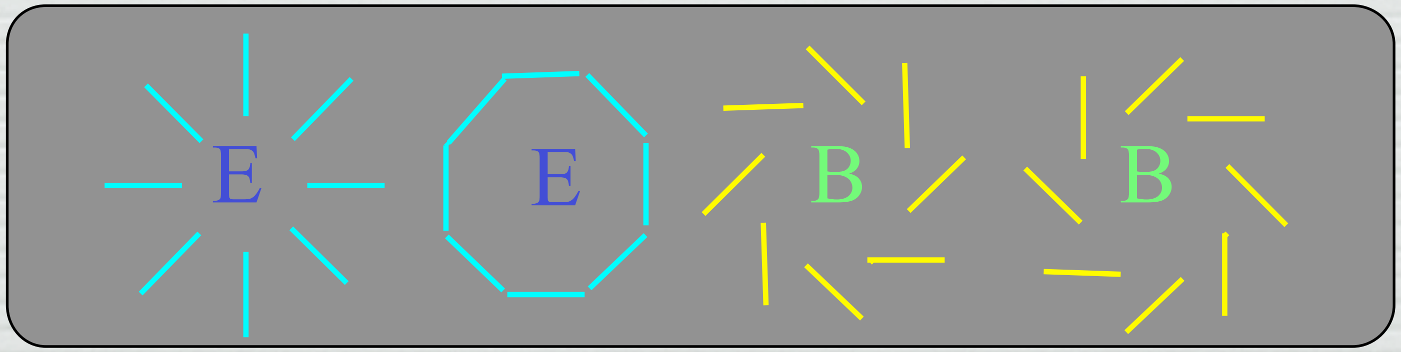
- Ionized plasma + quadrupole radiation field:
  - Thomson scattering ⇒ [linearly] polarized emission



- Unlike intensity, only generated when ionization fraction, $0<x<1$ (*i.e.*, during transition)
- *Scalar* perturbations: traces ~gradient of velocity
  - same initial conditions as temperature and density fluctuations
- *Tensor* perturbations: independent of density fluctuations
  - +,× patterns of quadrupoles (impossible to form via linear scalar perturbations)
  - at last-scattering, from primordial background of gravitational radiation, *predicted by inflation* (*cf. Senatore's lectures*)

Polarization: E/B

- 2-d (headless) vector field on a sphere
- Spin-2/tensor spherical harmonics
- grad/scalar/E + curl/pseudoscalar/B patterns

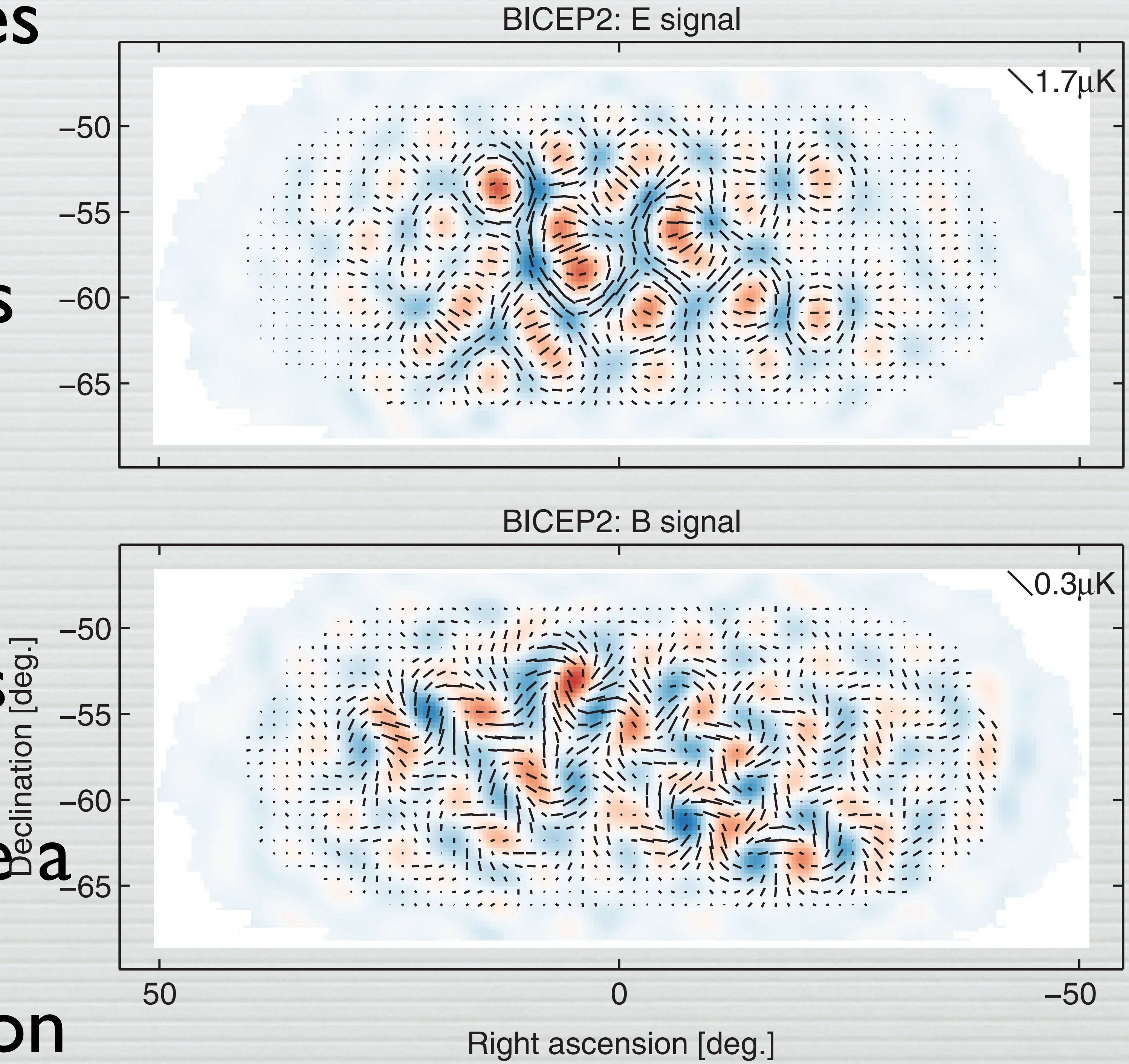


- NB. From polarization pattern ⇒ E/B decomposition requires integration (*non-local*) or differentiation (*noisy*)
  - *Lewis et al; Bunn et al; Smith & Zaldarriaga; Grain et al; Bowyer & AJ; …*
  - (data analysis problems)

Seljak & Zaldarriaga

B modes ⇒ gravitational waves?

- Everything generates E modes
- Everything *except* scalar perturbations generate B modes
- *If* we can rule out lensing, foregrounds and instrumental effects, B modes are a signature of gravitational radiation in the early Universe



Polarization: math

- Scalar and tensor modes are isotropic, parity-symmetric fields on the sky.
- T is a scalar, E is the "gradient" of a scalar, B is the "curl" of a pseudoscalar

$$Q(\hat{n}) = -\frac{1}{2}\sum_{lm}\left(a^E_{lm}\left[{}_2Y_{lm}(\hat{n}) + {}_{-2}Y_{lm}(\hat{n})\right] + i a^B_{lm}\left[{}_2Y_{lm}(\hat{n}) - {}_{-2}Y_{lm}(\hat{n})\right]\right)$$

$$U(\hat{n}) = -\frac{1}{2}\sum_{lm}\left(a^B_{lm}\left[{}_2Y_{lm}(\hat{n}) + {}_{-2}Y_{lm}(\hat{n})\right] + i a^E_{lm}\left[{}_2Y_{lm}(\hat{n}) - {}_{-2}Y_{lm}(\hat{n})\right]\right))$$

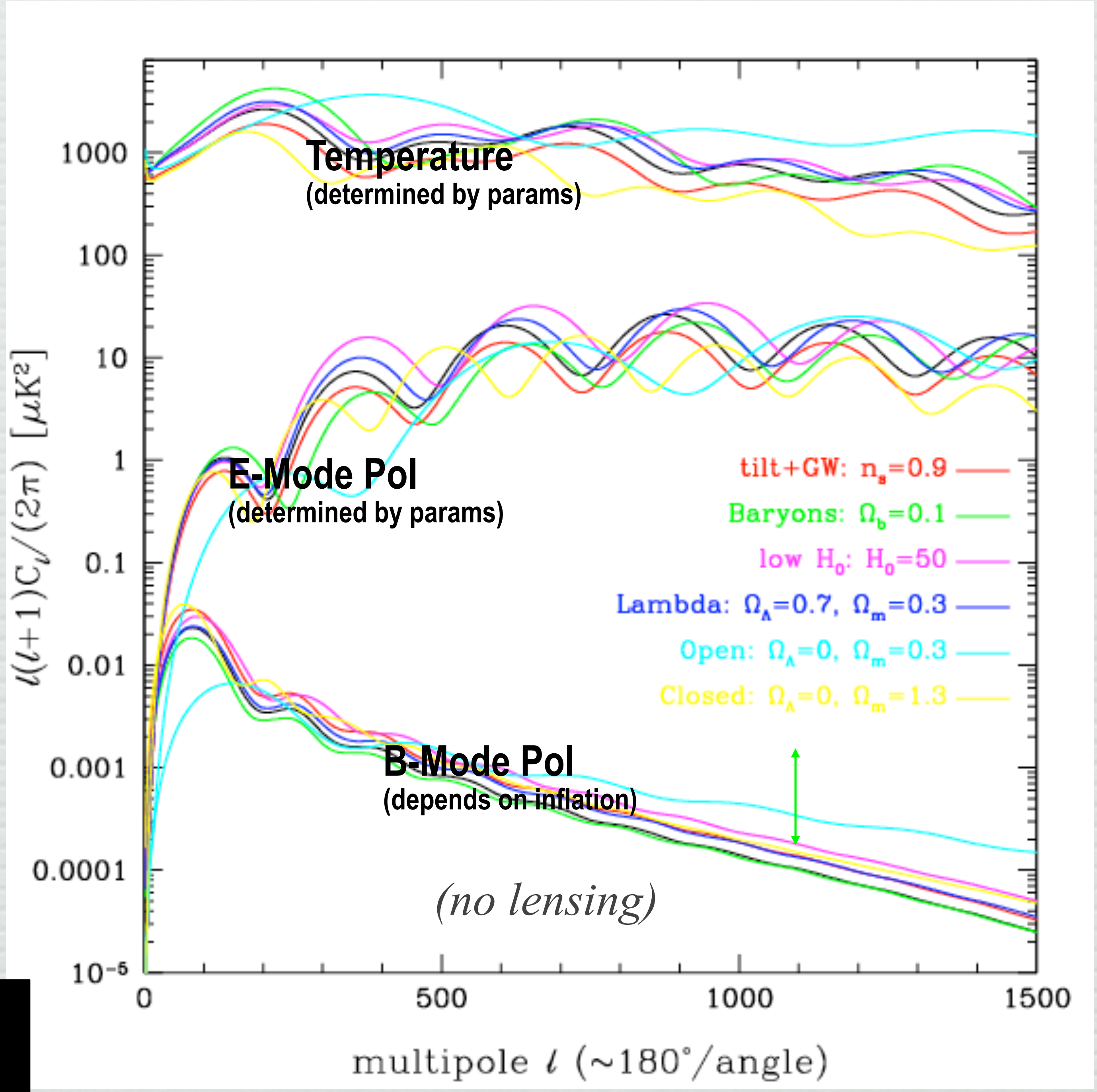
$$e(\hat{n}) = \sum_{lm}\sqrt{\frac{(l-2)!}{(l+2)!}}a^E_{lm}Y_{lm}(\hat{n}) \qquad b(\hat{n}) = \sum_{lm}\sqrt{\frac{(l-2)!}{(l+2)!}}a^B_{lm}Y_{lm}(\hat{n})$$

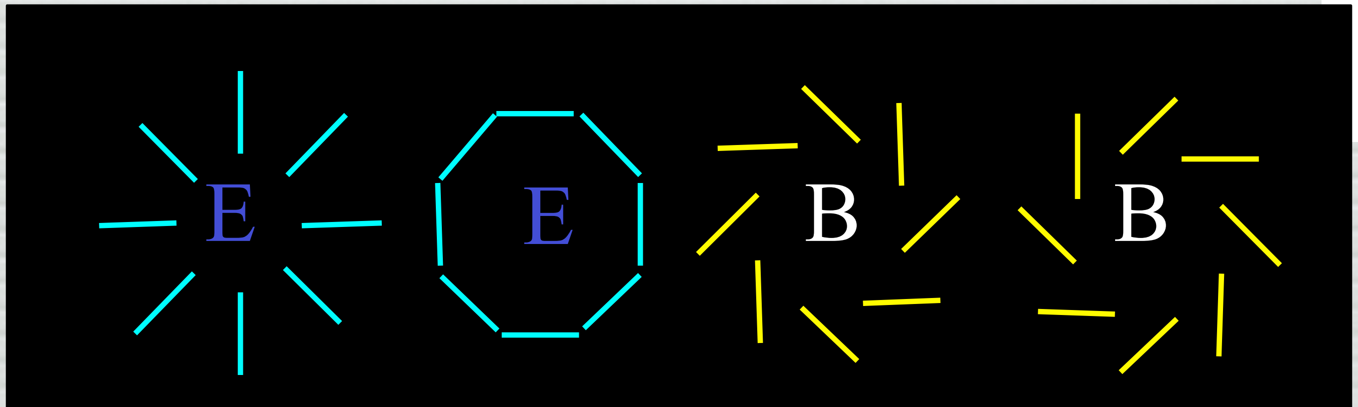
$$\nabla^4 e = -\frac{1}{2}[\bar{\eth}^2(Q+iU) + \eth^2(Q-iU)] \qquad \nabla^4 b = \frac{i}{2}[\bar{\eth}^2(Q+iU) - \eth^2(Q-iU)]$$

  - expect $\langle EB \rangle = \langle TB \rangle = 0$
  - try to measure $\langle TT \rangle$, $\langle BB \rangle$, $\langle EE \rangle$, $\langle TE \rangle$

The Polarization of the CMB

- Anisotropic radiation field at last scattering → polarization
  - "Grad" or *E* mode
  - Breaks degeneracies
  - New parameters:
    - reionization
- "Curl" or *B* sensitive to gravity waves
  - "Smoking gun" of inflation?
  - Very low amplitude
- Need better handle on systematics, and...
- Polarized foregrounds?

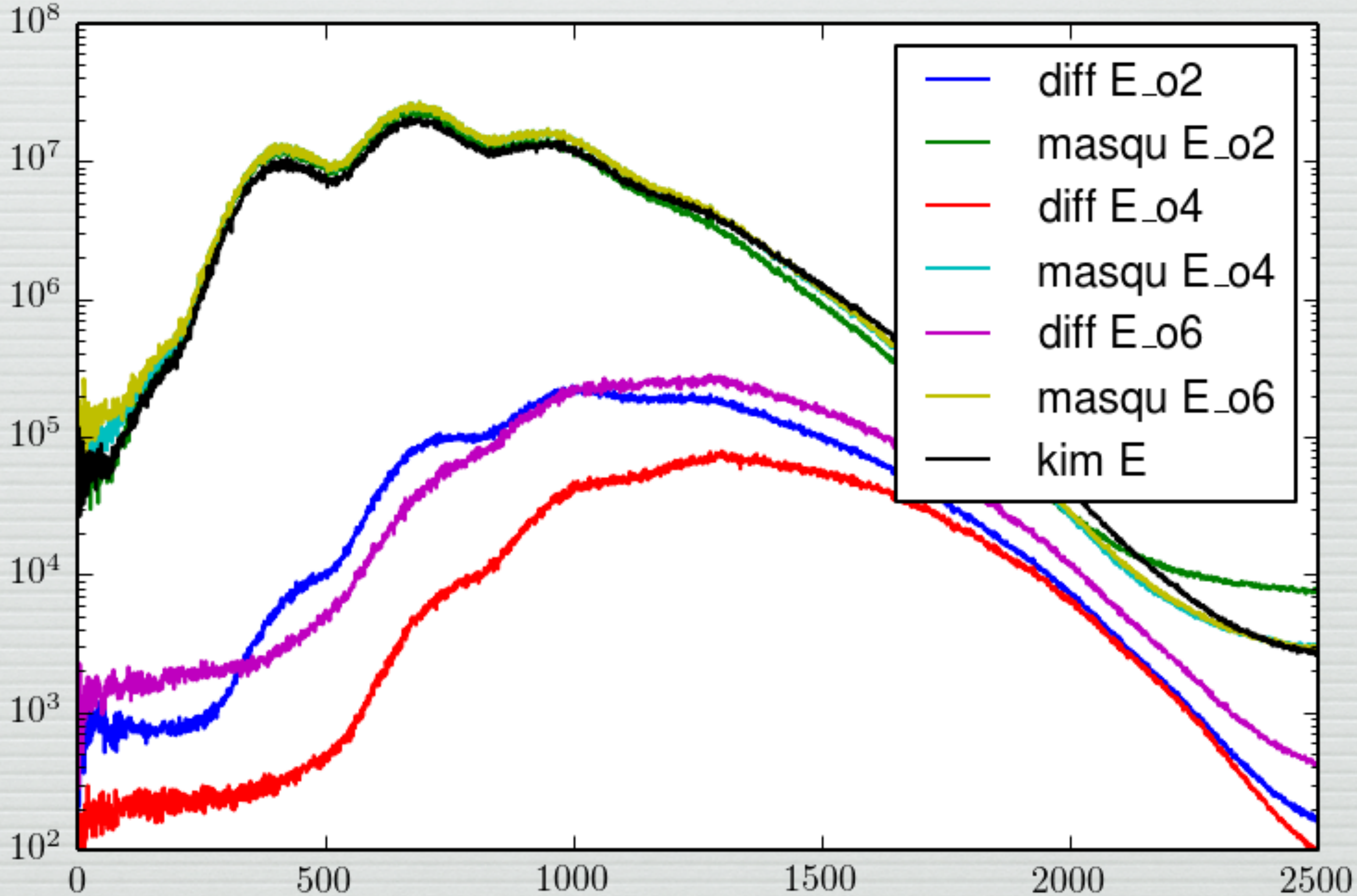




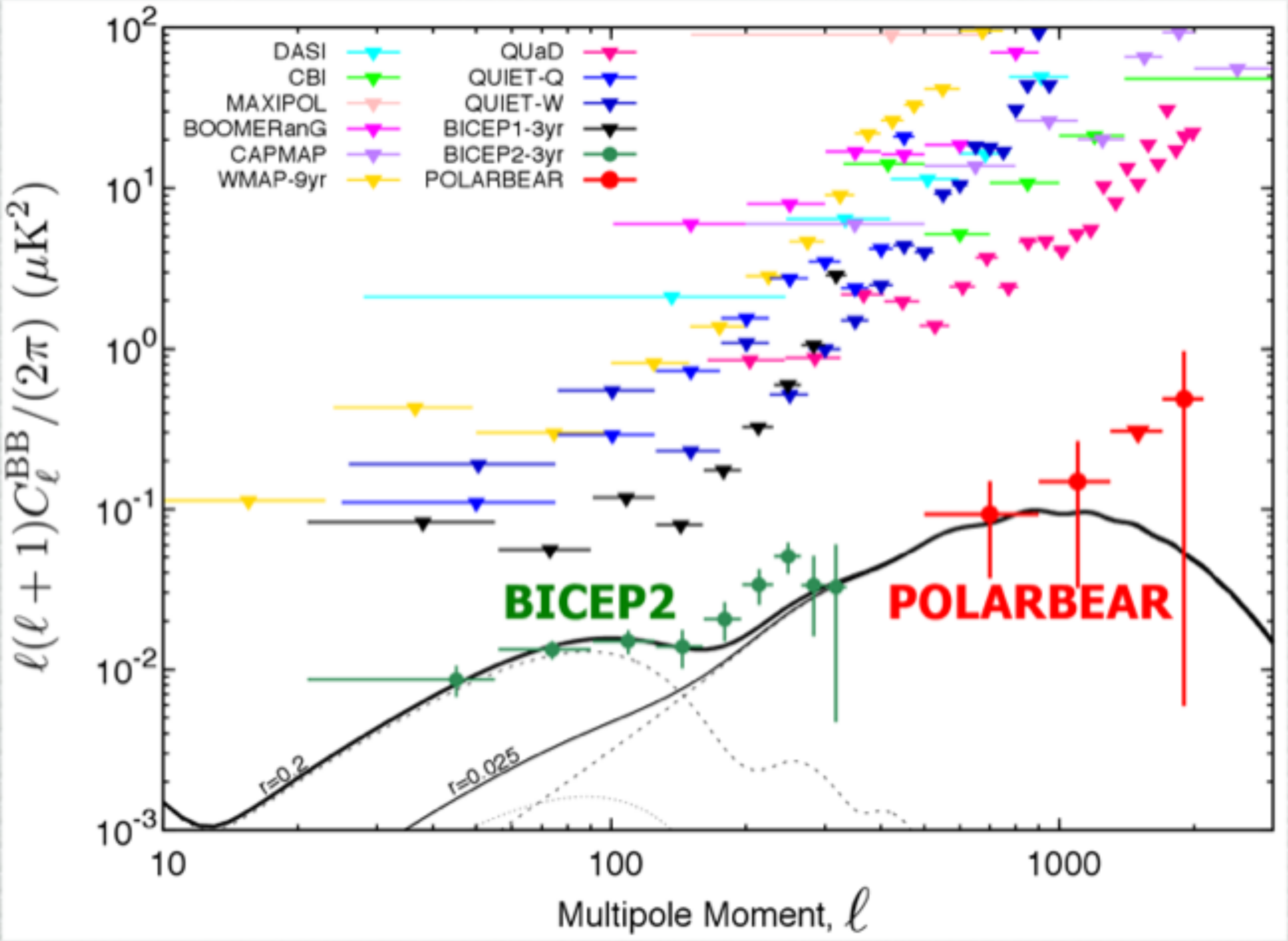
Polarization: data

- Formally the same problem
- $d_p \Rightarrow (i,q,u)_p = d_{i,p} = d_q \qquad \langle d_q\, d_{q'} \rangle = N_{qq'} + S_{qq'}$
- Correlation matrix become combination of ⟨TT⟩, ⟨TE⟩, ⟨EE⟩, ⟨BB⟩ (other combinations usu. vanish)
  - Sphere enters in this calculation through ${}_{\pm 2}Y_{\ell m}$, P_ℓ, &c
- Sometimes useful to actually transform from Q/U→E/B
  - "trivial" on full (pixelized) QU sky with no noise.
  - realistic case more complicated (e.g., Smith & Zaldariagga)
    - harmonic analyses (e.g., Kim 2010)
    - real-space finite differences (Bowyer, AHJ, Novikov 2011)
    - still have to "undo" Laplacian to get to true e/b scalars
      - inherently nonlocal

Polarization: E/B Separation



Polarization: State of the Art



non-Gaussianity: f_{NL}

- Heuristically $\phi = \phi_G + f_{\rm NL}(\phi_G^2 - \langle\phi_G^2\rangle)$
  for a Gaussian ϕ_G (e.g., multi-field inflation)
  - This is the (spatially) local model for non-Gaussianity
  - Induces specific 3-d correlations

    $$\begin{aligned}\langle\phi\phi\phi\rangle &\sim 3f_{\rm NL}\left(\langle\phi_G\phi_G\phi_G\phi_G\rangle - \langle\phi_G\phi_G\rangle\langle\phi_G\phi_G\rangle\right) + O(f_{\rm NL}^2)\\ &\sim 6f_{\rm NL}\langle\phi_G\phi_G\rangle\langle\phi_G\phi_G\rangle + O(f_{\rm NL}^2)\end{aligned}$$

    and hence 2-d correlations in the CMB
  - Corresponds to Fourier bispectrum $B(k_1, k_2, k_3)$ which peaks in squeezed case $k_1 \ll k_2 \simeq k_3$
    - modulate small-scale structure by large-scale modes
      - cf. galaxy bias
  - More generally, consider other shapes (e.g., equilateral) motivated by specific theories

non-Gaussianity on the sphere

- Consider the “connected” n-point functions
  $$\langle a_{\ell_1 m_1} a_{\ell_2 m_2} \cdots a_{\ell_n m_n} \rangle_c$$
  isotropy gives a constraint on the $\{\ell_i m_i\}$
  - equivalent to Euclidean $\boldsymbol{k}_1+\boldsymbol{k}_2+\cdots+\boldsymbol{k}_n = 0$
  - (seen in vanishing of 3-j, 6-j, Wigner D, … functions)

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If the CMB sky is rotationally invariant, the angular bispectrum can be factorized as follows:

$$\langle a_{\ell_1 m_1} a_{\ell_2 m_2} a_{\ell_3 m_3} \rangle = \mathcal{G}^{\ell_1 \ell_2 \ell_3}_{m_1 m_2 m_3} \, b_{\ell_1 \ell_2 \ell_3} \,, \qquad (24)$$

where $b_{\ell_1 \ell_2 \ell_3}$ is the so called *reduced bispectrum*, and $\mathcal{G}^{\ell_1 \ell_2 \ell_3}_{m_1 m_2 m_3}$ is the Gaunt integral, defined as:

$$\begin{aligned} \mathcal{G}^{\ell_1 \ell_2 \ell_3}_{m_1 m_2 m_3} &\equiv \int Y_{\ell_1 m_1}(\hat{\boldsymbol{n}})\, Y_{\ell_2 m_2}(\hat{\boldsymbol{n}})\, Y_{\ell_3 m_3}(\hat{\boldsymbol{n}})\, d^2 \hat{\boldsymbol{n}} \\ &= h_{\ell_1 \ell_2 \ell_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}, \end{aligned} \qquad (25)$$

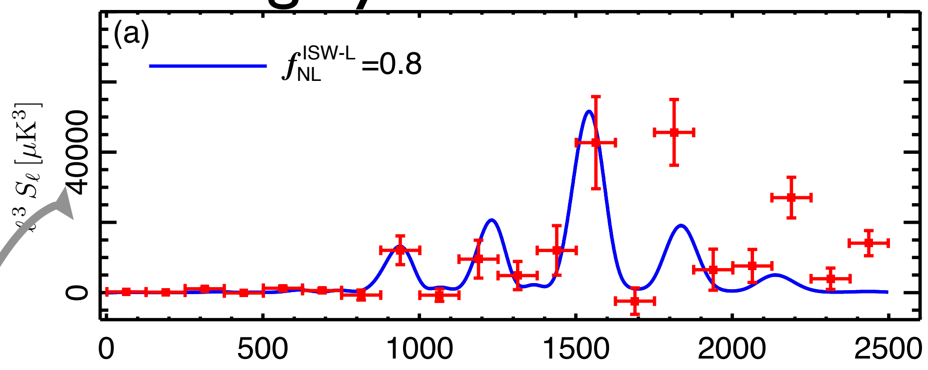
where $h_{\ell_1 \ell_2 \ell_3}$ is a geometrical factor,

$$h_{\ell_1 \ell_2 \ell_3} = \sqrt{\frac{(2\ell_1 + 1)(2\ell_2 + 1)(2\ell_3 + 1)}{4\pi}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (26)$$

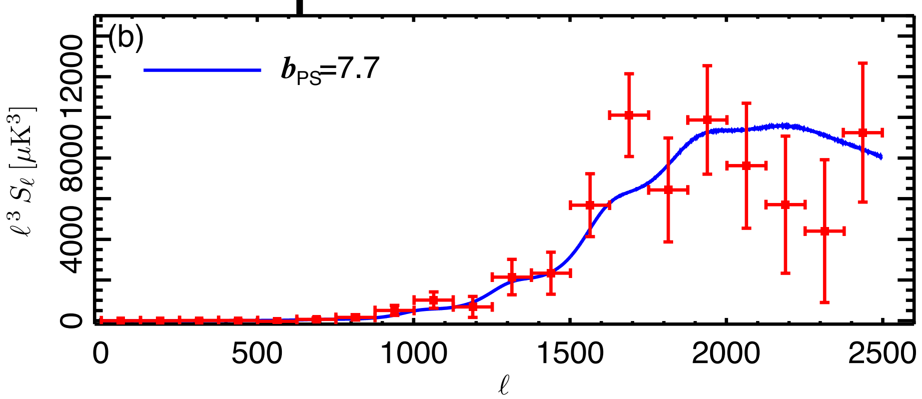
- Spherical manifold also enters in choice of an estimator for the n-point function (or f_{NL}) — usually start from estimates of harmonic coefficients $a_{\ell m}$

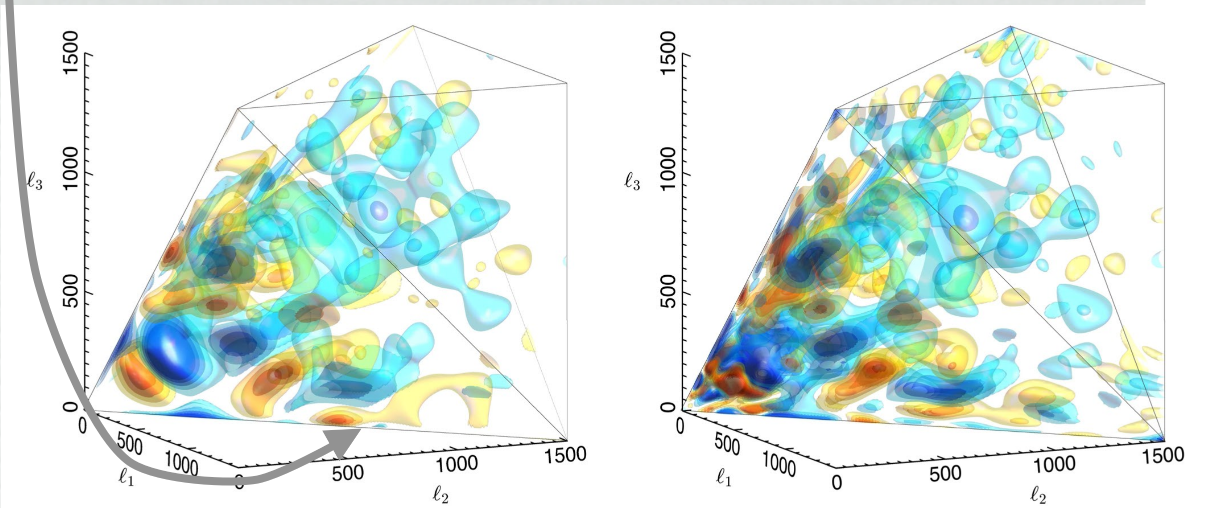
Non-Gaussianity

We see
lensing by cosmic structures

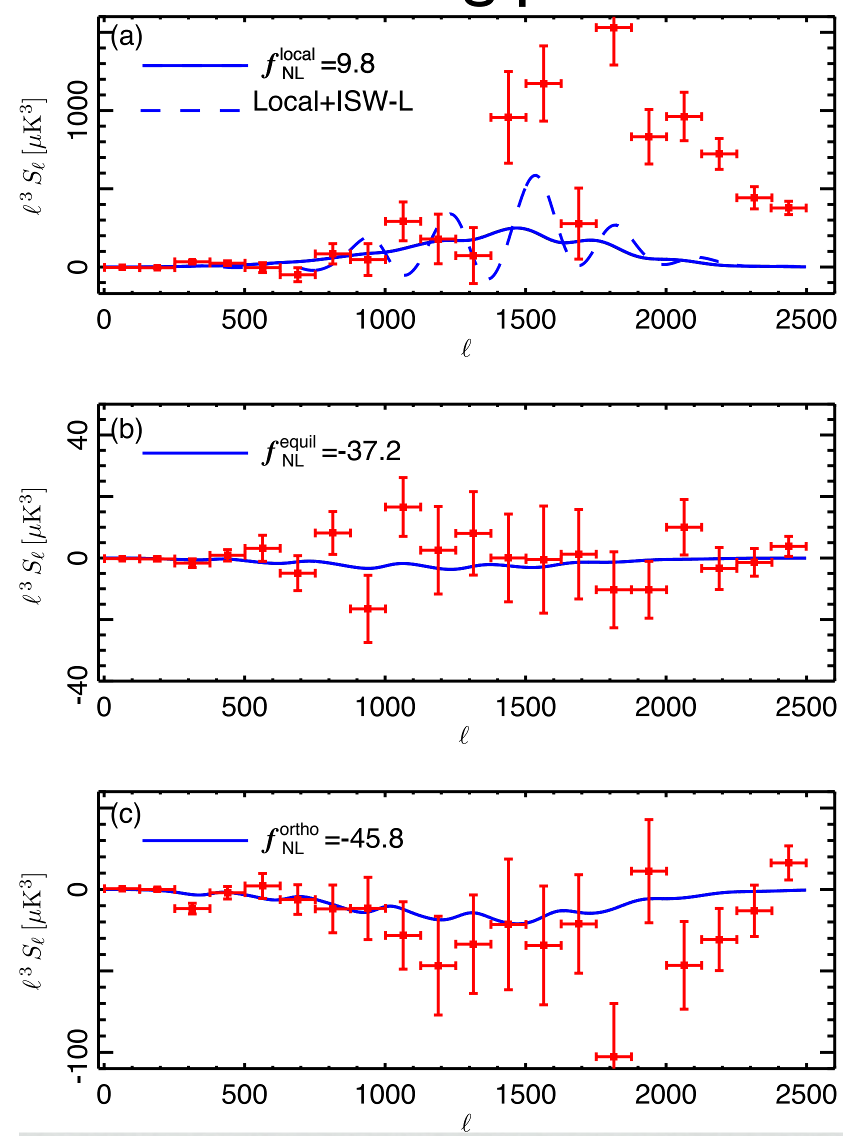


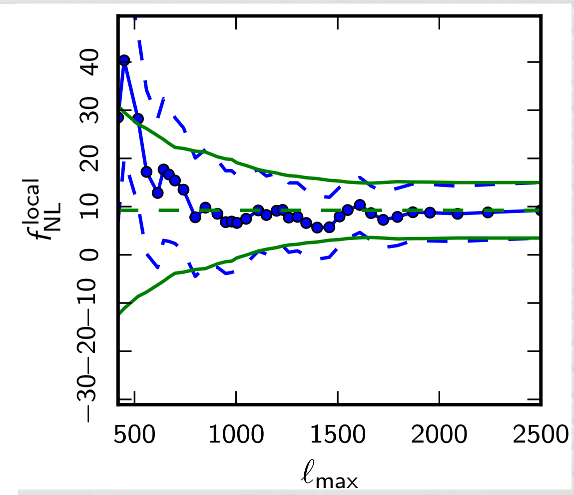
…and point sources…





…but nothing primordial



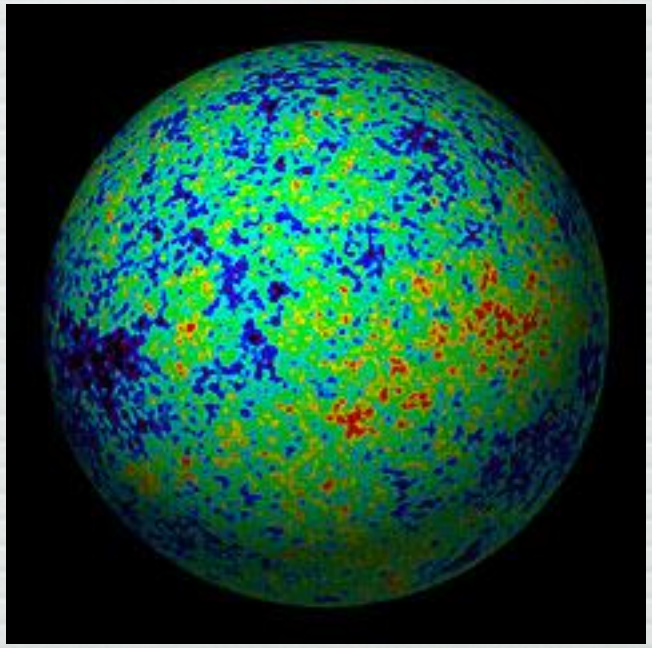


$$f_{NL}^{local} = 2.7 \pm 5.8$$

$$f_{NL}^{equil} = -42 \pm 75$$

$$f_{NL}^{ortho} = -25 \pm 39$$

Conclusions



- The CMB lives on the sphere
- Simple description of signal: C_ℓ
- Noise, masking, etc break the symmetries
- Brute force statistical approaches (optimal, Bayesian) don't really take advantage of the available mathematics
  - But too expensive in practice (on large datasets)
- realistic (approximate) methods (pseudo-spectra, component separation, E/B separation, …) have all taken advantage of the spherical signal