Flaglets for studying the large-scale structure of the Universe

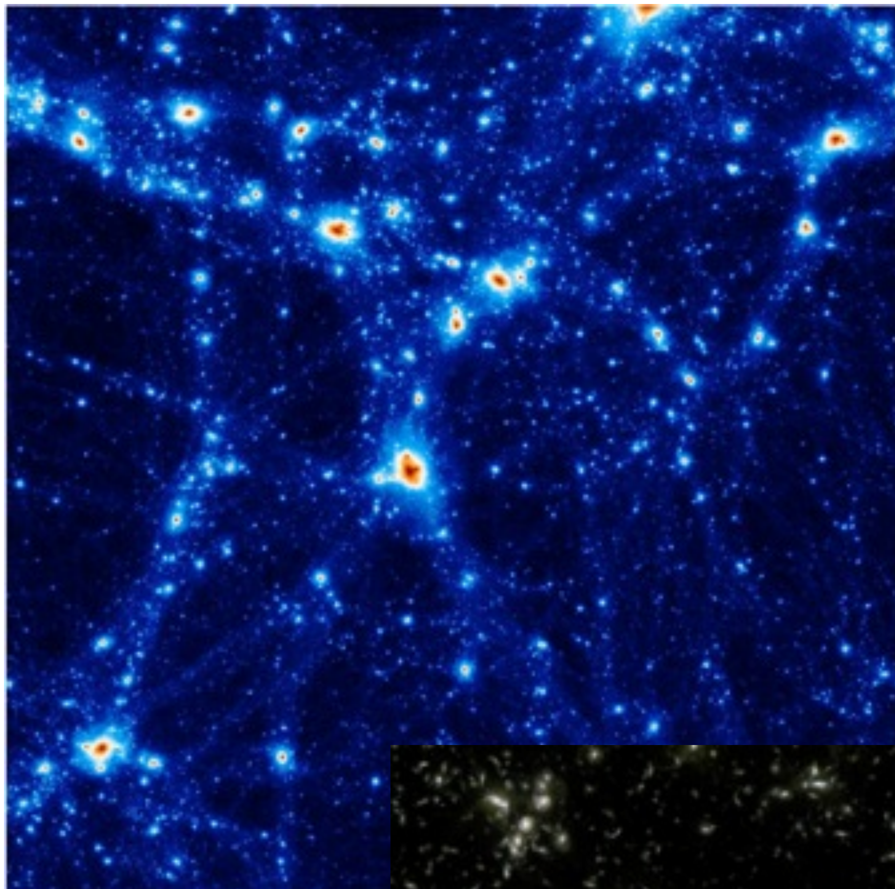
Boris Leistedt
Cosmology Group, University College London

Based on arXiv:1205.0792, 1308.5480, 1308.5406
with Hiranya Peiris, Jason McEwen, Martin Büttner

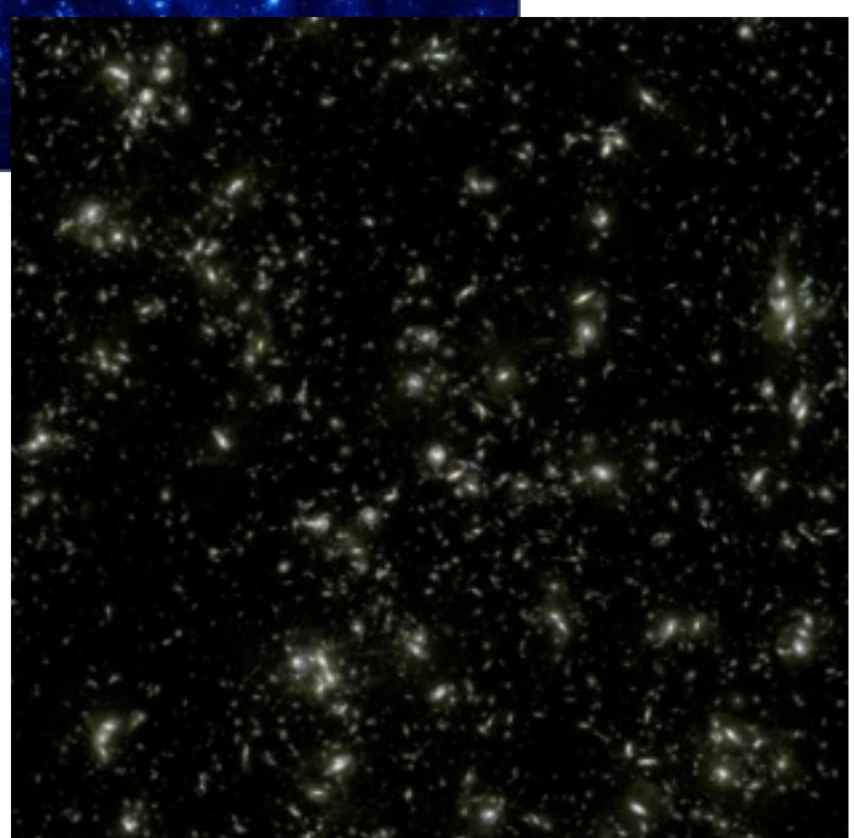
Roadmap

- **Galaxy surveys & data on the ball**
- Fourier-Laguerre transform on the ball
- Flaglet transform on the ball
- Spin directional wavelets on the sphere & ball

Cosmology with galaxy surveys



Dark matter
=invisible

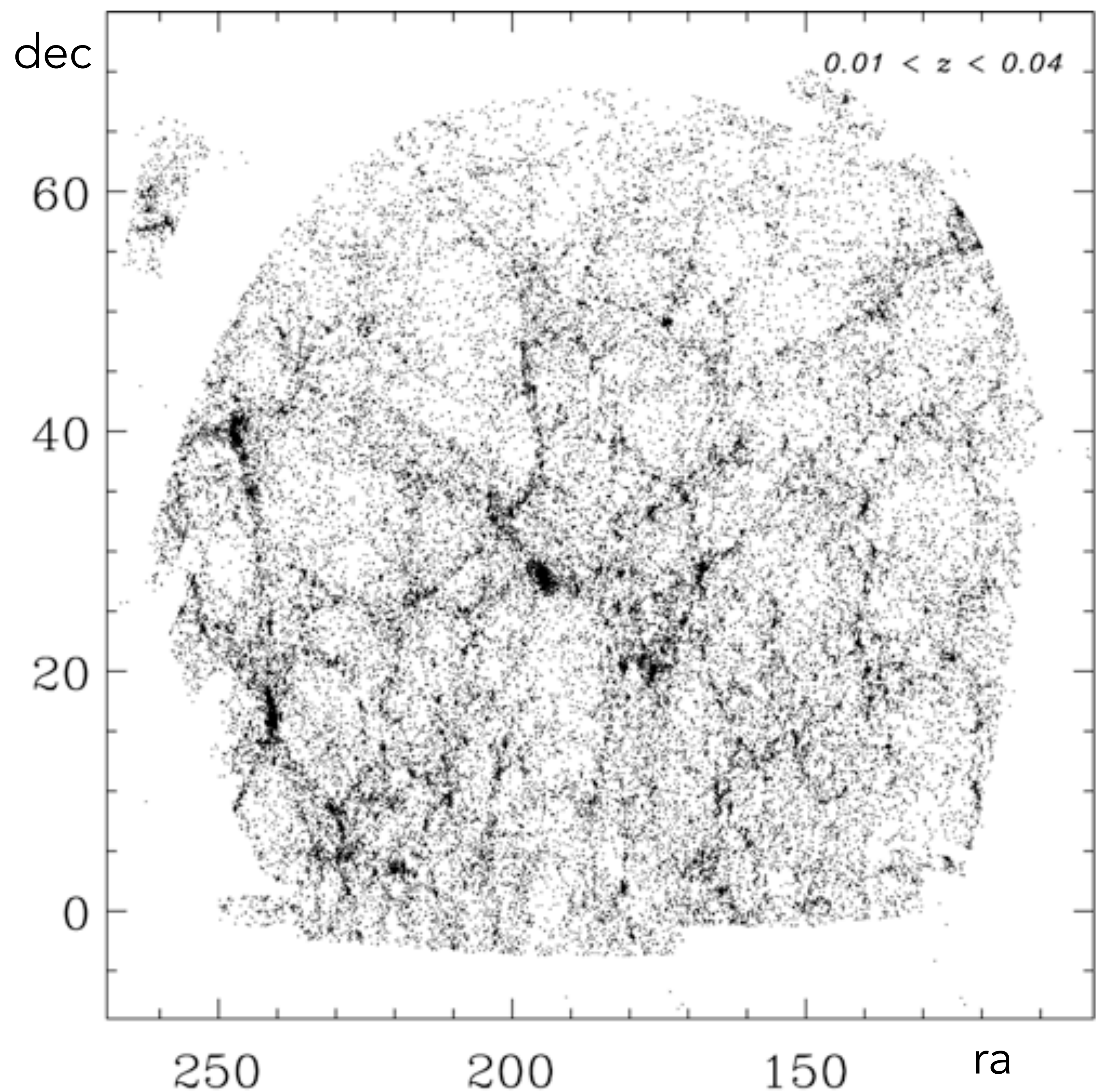


Galaxies
=visible

- Nature and properties of dark matter, dark energy?
- GR or modified gravity?
- Origin of structure? Inflation?
- Signatures imprinted in the large-scale structure

The large-scale structure of the Universe

Redshift slices of SDSS DR7

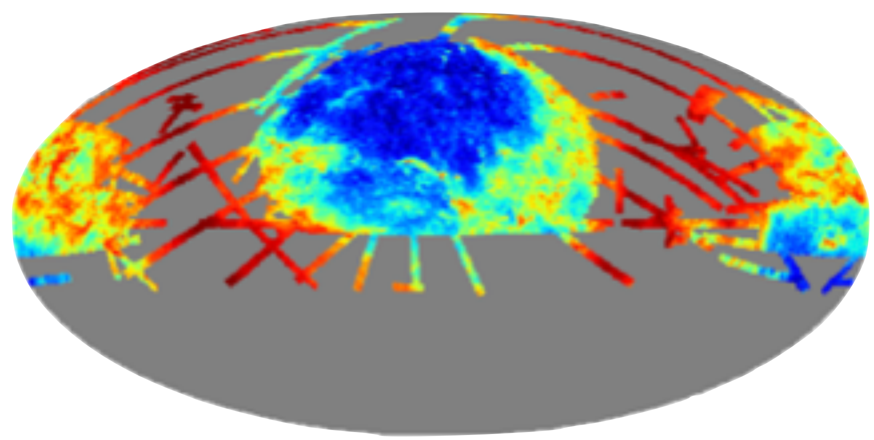


- Angle on the sky + redshift = **3D position**
- **3D cosmic web :** filaments, walls, voids due to hierarchical structure formation

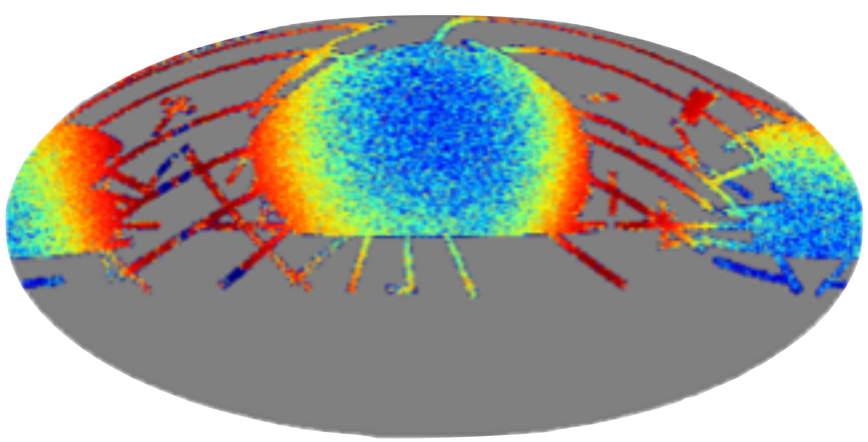
Exploiting LSS data ***is*** complicated

- Photometric surveys (DES, Euclid, LSST): new challenges
- Photo-z errors, spatially varying systematics / depth
- Complicated geometry / selection functions

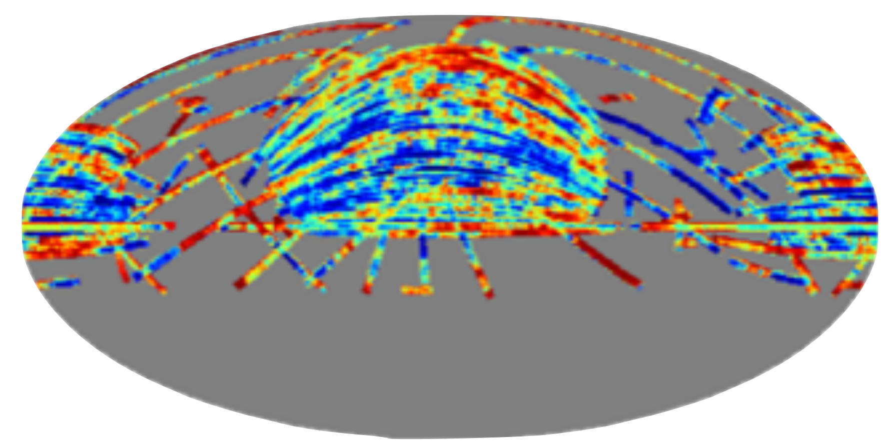
dust



stars

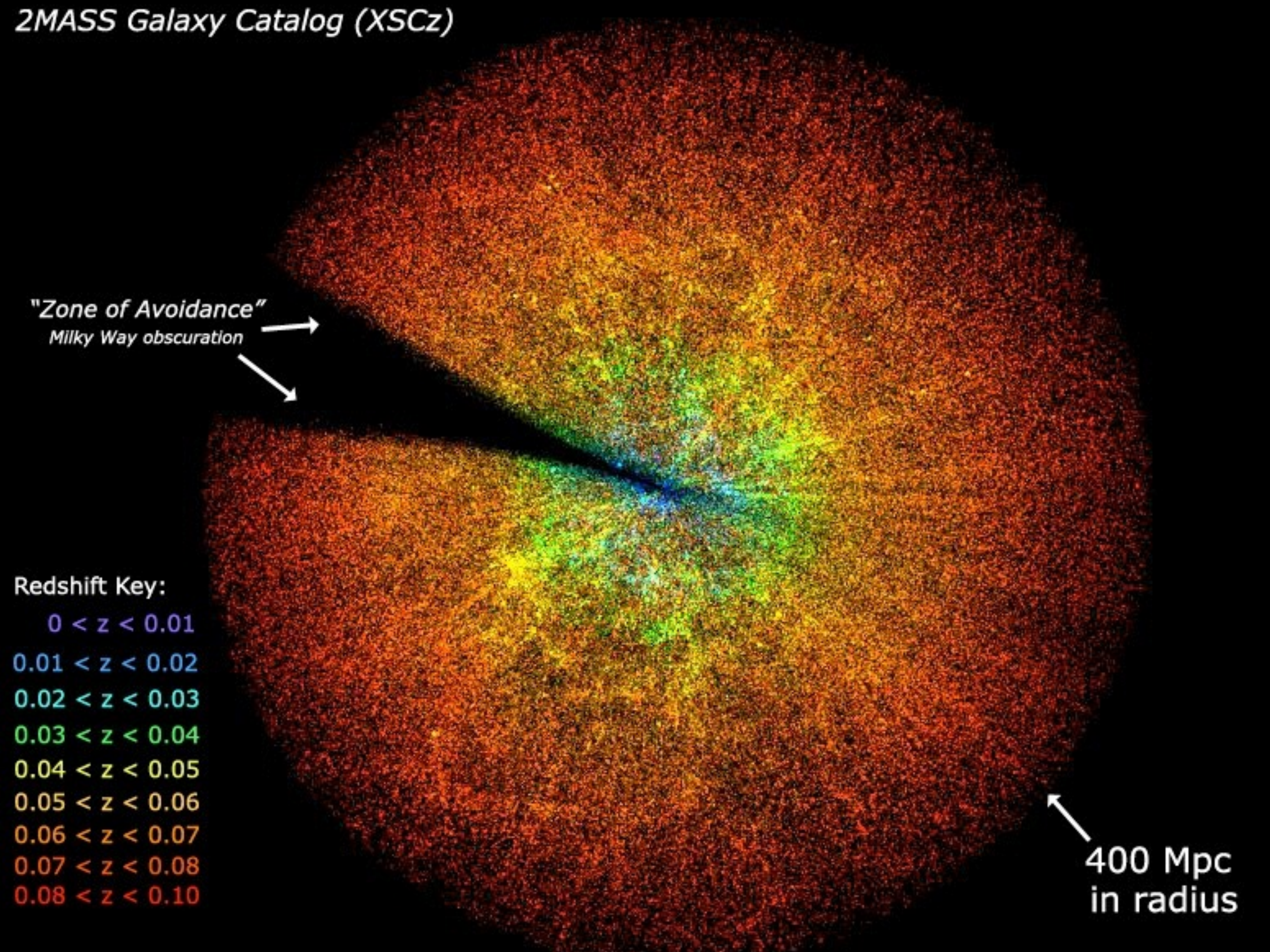


seeing

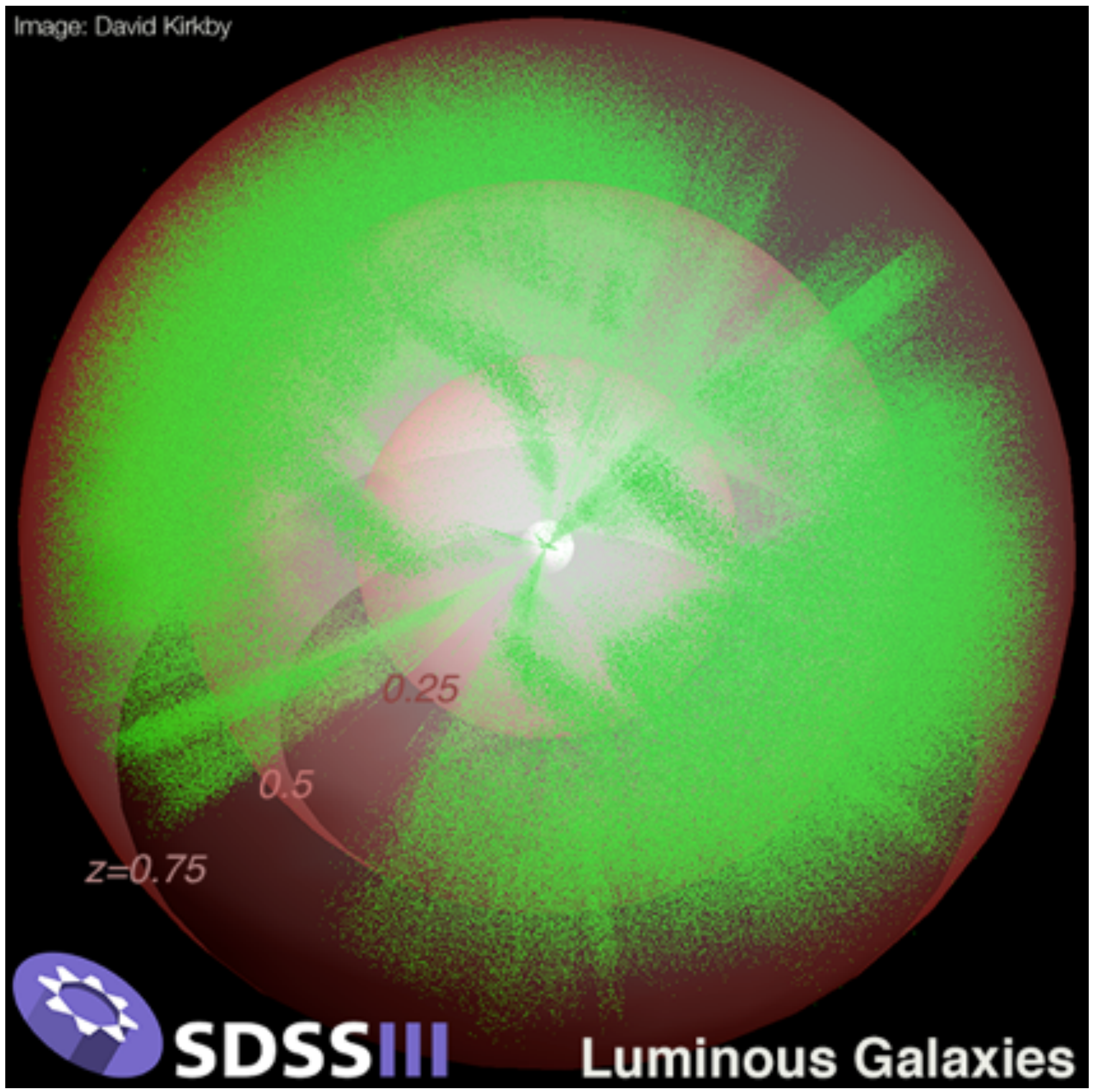


Systematics is the new frontier

Appropriate methods are essential


2MASS Galaxy Catalog (XSCz)
"Zone of Avoidance"
Milky Way obscuration
Redshift Key:
0 < z < 0.01
0.01 < z < 0.02
0.02 < z < 0.03
0.03 < z < 0.04
0.04 < z < 0.05
0.05 < z < 0.06
0.06 < z < 0.07
0.07 < z < 0.08
0.08 < z < 0.10
400 Mpc
in radius

Wish list for novel 3D transforms



- 3D spherical measure

$$\mathrm{d}^3\vec{r} = r^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi \mathrm{d}r$$

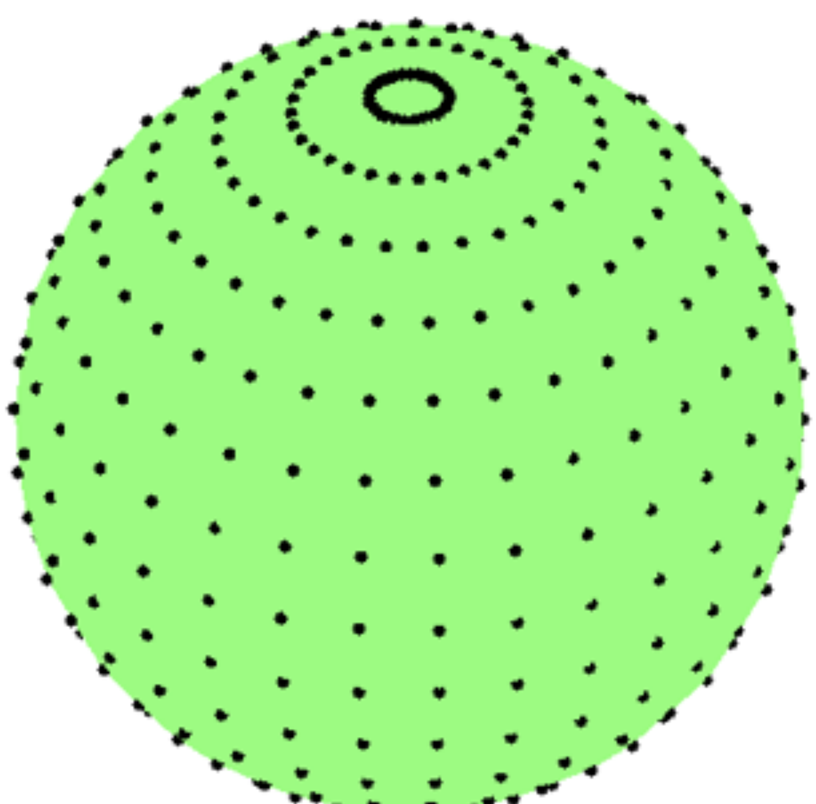
- Separable (data on $\mathbb{S}^2 \times \mathbb{R}^+$ rather than $\mathbb{R}^3$)
- Meaningful translation, rotation operators
- Theoretically exact / sampling theorem
- Relate to Fourier-Bessel

Roadmap

- Galaxy surveys & data on the ball
- **Fourier-Laguerre transform on the ball**
- Flaglet transform on the ball
- Spin directional wavelets on the sphere & ball

Exact spherical harmonic transform

- Spherical harmonics: $f(\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m} \boxed{Y_{\ell m}(\theta,\phi)}$
- MW sampling theorem : band-limited at L $\Longleftrightarrow$ information captured in $\mathrm{N_{pix}} \sim 2L^2$ samples



$\Longrightarrow$ integrals discretised without any approximation

$\Longrightarrow$ theoretically exact transform

McEwen & Wiaux (2011)

Exact spherical Laguerre transform

- Basis functions on $\mathbb{R}^+$

$$K_p(r) \equiv \sqrt{\frac{p!}{(p+2)!}} \frac{e^{-r/2\tau}}{\sqrt{\tau^3}} L_p^{(2)}\left(\frac{r}{\tau}\right)$$

- Exact transform:

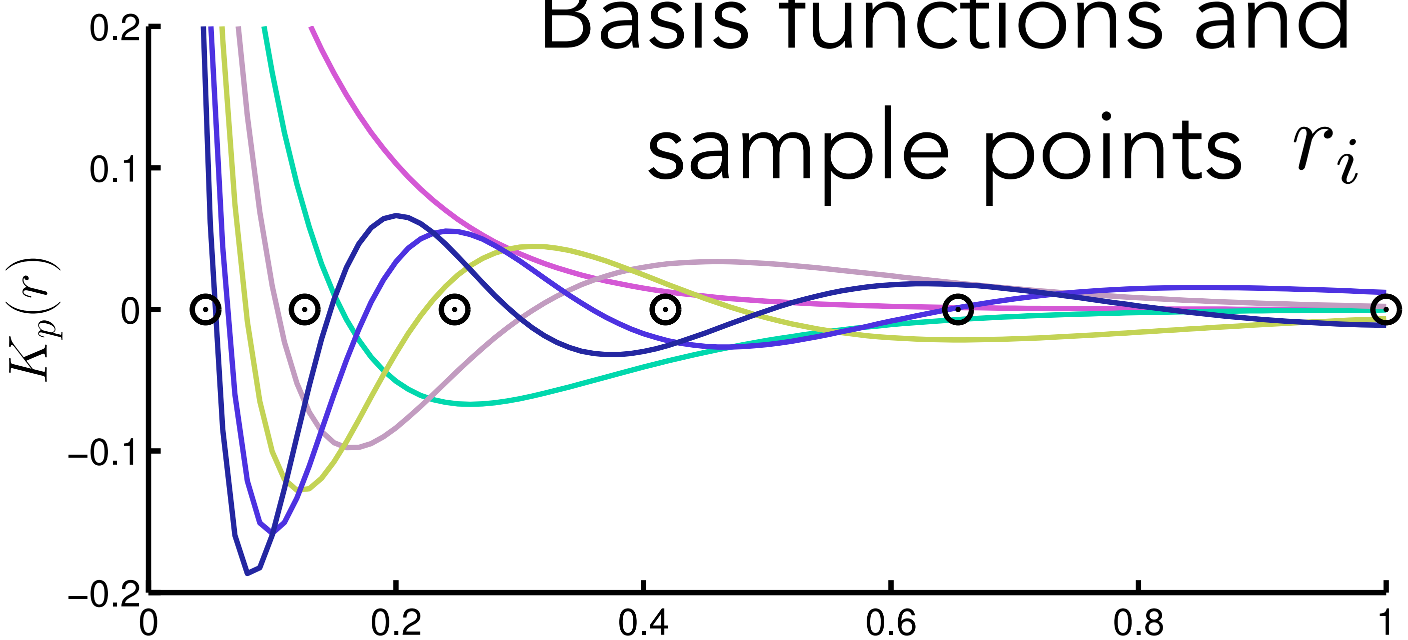
$$f(r) = \sum_{p=0}^{P-1} f_p K_p(r)$$

$$f_p = \sum_{i=0}^{P-1} w_i f(r_i) K_p(r_i)$$

- P samples on [0, R]

P = 6

Basis functions and sample points r_i



Leistedt & McEwen (2012)

The Fourier-Laguerre transform

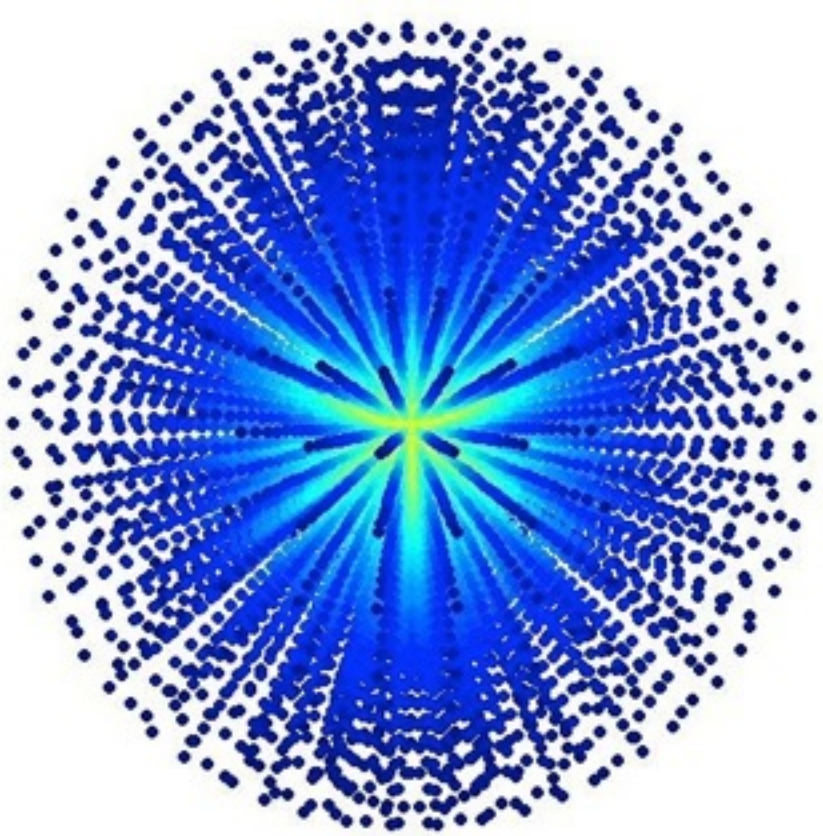
- Basis on $\mathbb{B}^3 = \mathbb{S}^2 \times \mathbb{R}^+$ with measure $\mathrm{d}^3\vec{r} = r^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi \mathrm{d}r$

$$Z_{\ell m p}(\vec{r}) = K_p(r) Y_{\ell m}(\theta, \phi) \qquad \vec{r} = (r, \theta, \phi)$$

- For band limited signals,

$$f_{\ell m p} = 0, \forall \ell \geq L, \forall p \geq P$$

f reconstructed on
$f_{\ell m p}$ calculated from } $\sim 2PL^2$ samples



Leistedt & McEwen (2012)

Connection to Fourier-Bessel analysis

- Basis functions: $Z'_{\ell m}(k;\vec{r}) = Y_{\ell m}(\theta,\phi) j_\ell(kr)$
- No sampling theorem for $\int_{\mathbb{R}+} f(r) j_\ell(kr) r^2 \mathrm{d}r$
- BUT can be calculated (exactly!) from Fourier-Laguerre:

$$\boxed{\tilde{f}_{\ell m}(k) = \sqrt{\frac{2}{\pi}} \sum_p f_{\ell m p} j_{\ell p}(k)} \quad \text{with} \quad j_{\ell p}(k) \equiv \langle K_p | j_\ell \rangle$$

finite sum if band-limited

has analytical formula

- Exact evaluation of Fourier-Bessel transform!

Leistedt & McEwen (2012)

Roadmap

- Galaxy surveys & data on the ball
- Fourier-Laguerre transform on the ball
- **Flaglet transform on the ball**
- Spin directional wavelets on the sphere & ball

Flaglet transform

Projection:

$$W^{\Phi}(\vec{r}) \equiv (f \star \Phi)(\vec{r}) = \langle f | \mathcal{T}_r \mathcal{R}_{(\theta,\phi)} \Phi \rangle$$

$$\boxed{W^{\Psi^{jj'}}(\vec{r}) \equiv (f \star \Psi^{jj'})(\vec{r}) = \langle f | \mathcal{T}_r \mathcal{R}_{(\theta,\phi)} \Psi^{jj'} \rangle}$$

Reconstruction:

$$f(r,\theta,\phi) = \int_{\mathbb{B}^3} W^{\Phi}(\vec{s})(\mathcal{T}_r \mathcal{R}_{(\theta,\phi)} \Phi)(\vec{s}) \mathrm{d}^3\vec{s} + \sum_{jj'} \int_{\mathbb{B}^3} W^{\Psi^{jj'}}(\vec{s})(\mathcal{T}_r \mathcal{R}_{(\theta,\phi)} \Psi^{jj'})(\vec{s}) \mathrm{d}^3\vec{s}$$

- Uses translation, rotation, convolution operators on $\mathbb{B}^3$
- **Done in harmonic space thanks to sampling theorem**

Leistedt & McEwen (2012)

Convolution on the ball

- 3D operator: $\mathcal{T}_{\vec{r}} \equiv \mathcal{T}_r \mathcal{R}_{(\theta,\phi)}$
  radial translation + SO(2) rotation

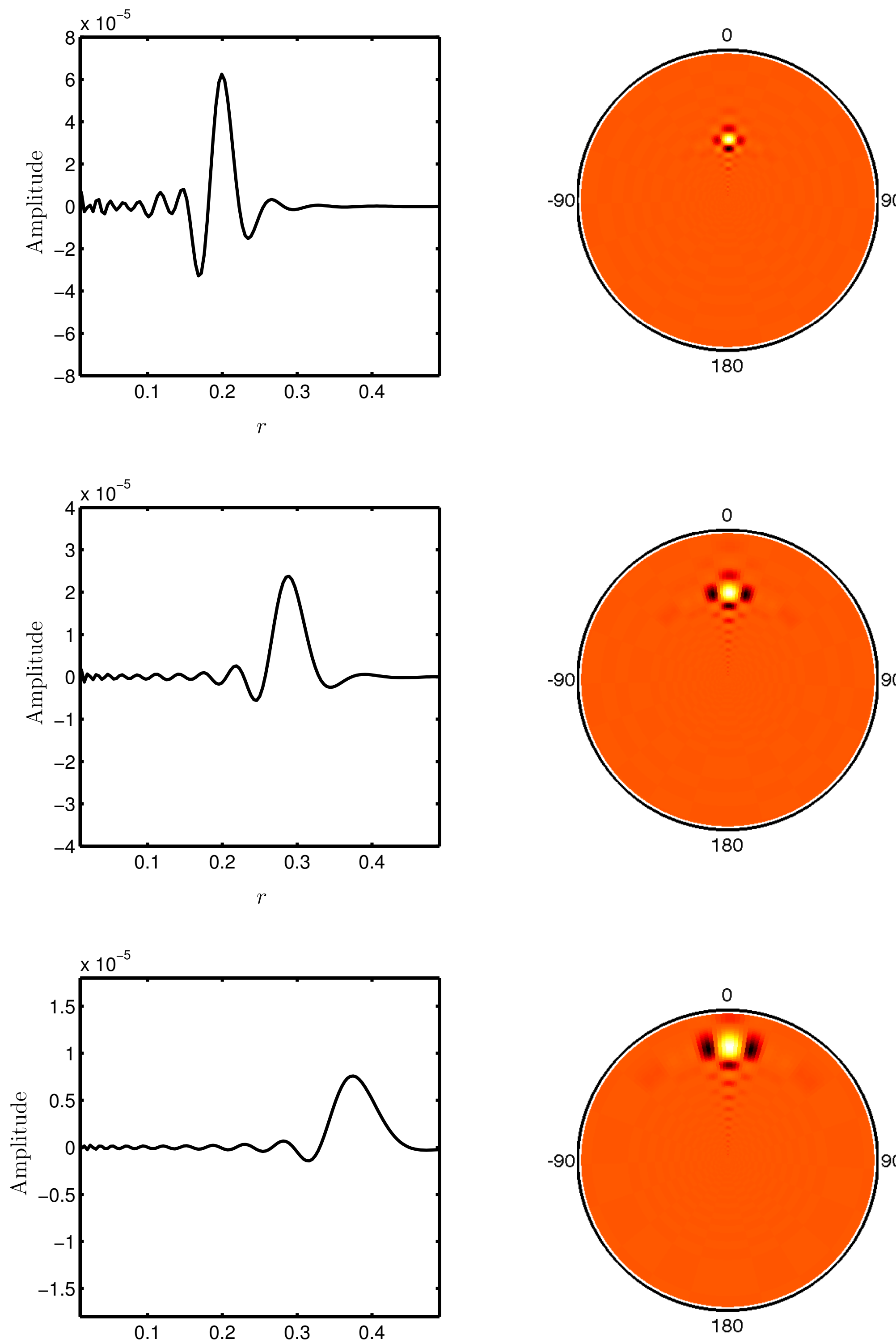
- Convolution on the ball:

$$(f \star \Psi^{jj'})(\vec{r}) = \langle f | \mathcal{T}_{\vec{r}} \rangle_{\mathbb{B}^3}$$
$$= \int_{\mathbb{B}^3} f(\vec{s})(\mathcal{T}_{\vec{r}} \Psi^{jj'})(\vec{s}) d\vec{s}$$

- Axisymmetric wavelets:

$$(f \star \Psi^{jj'})_{\ell m p} = \sqrt{\frac{4\pi}{2\ell+1}} f_{\ell m p} \Psi^{*}_{\ell 0 p}$$

Translated flaglets

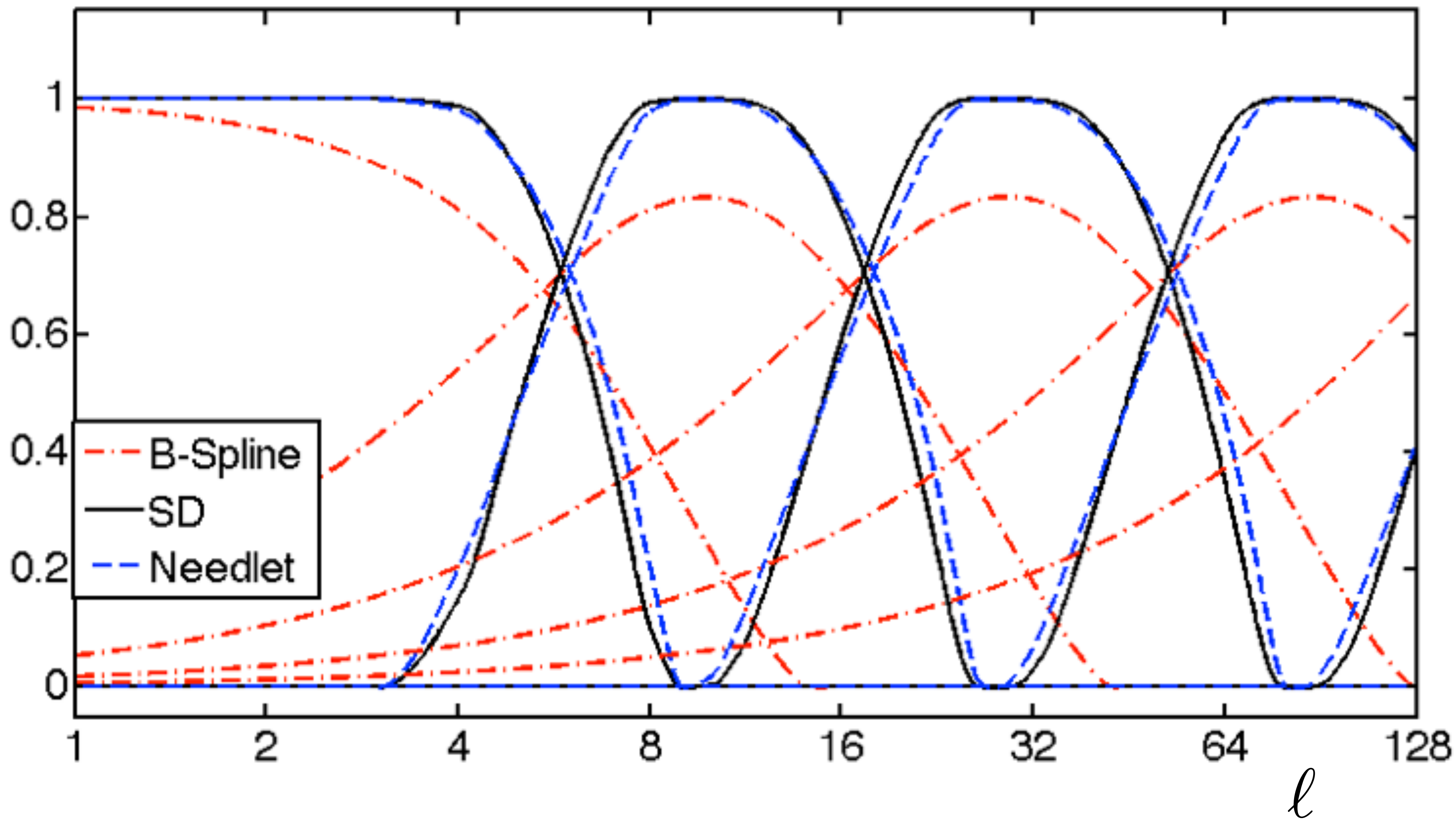


Scale-discretised wavelets on the sphere

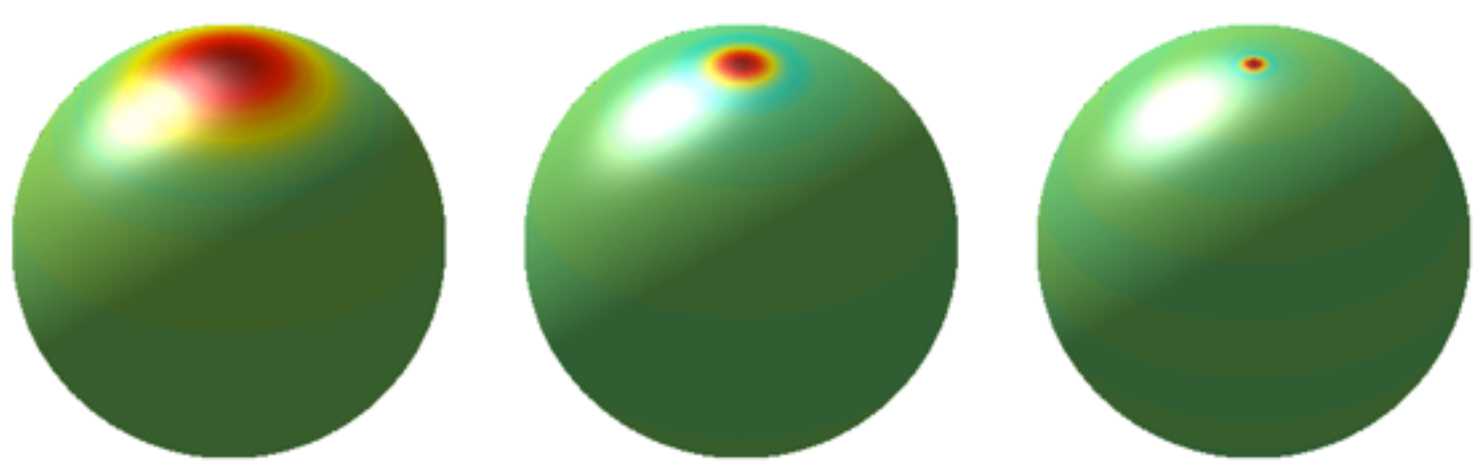
Wiaux et al (2009)

Tilling of the harmonic line

$$\Psi_\ell^j = \kappa^j(\ell)$$



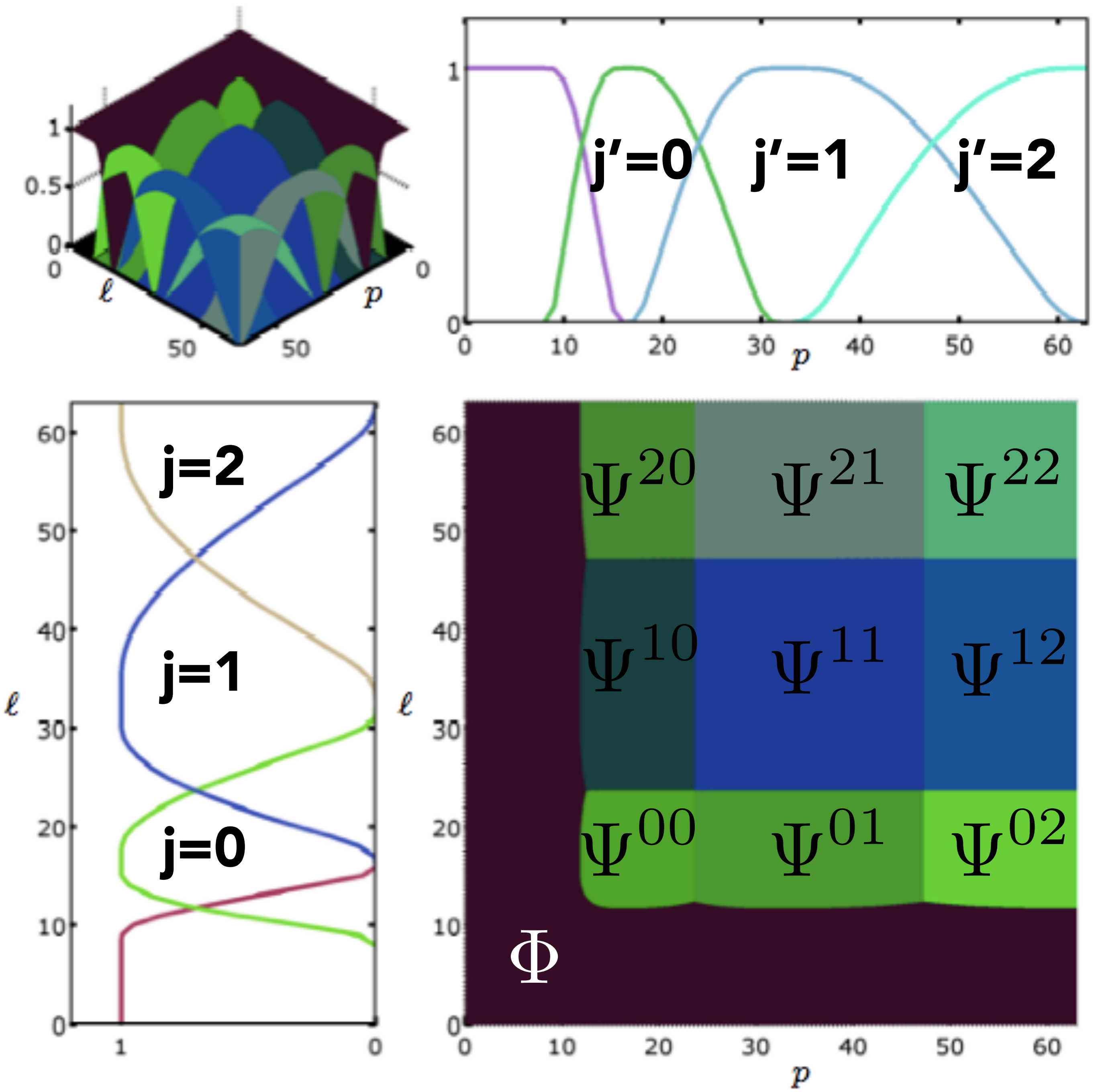
Resulting wavelets (***axisymmetric***)



Tiling of the Fourier-Laguerre space

Tilling of the radial harmonic line

Tilling of the angular harmonic line



Defining

$$\Psi^{jj'}_{\ell m p}$$

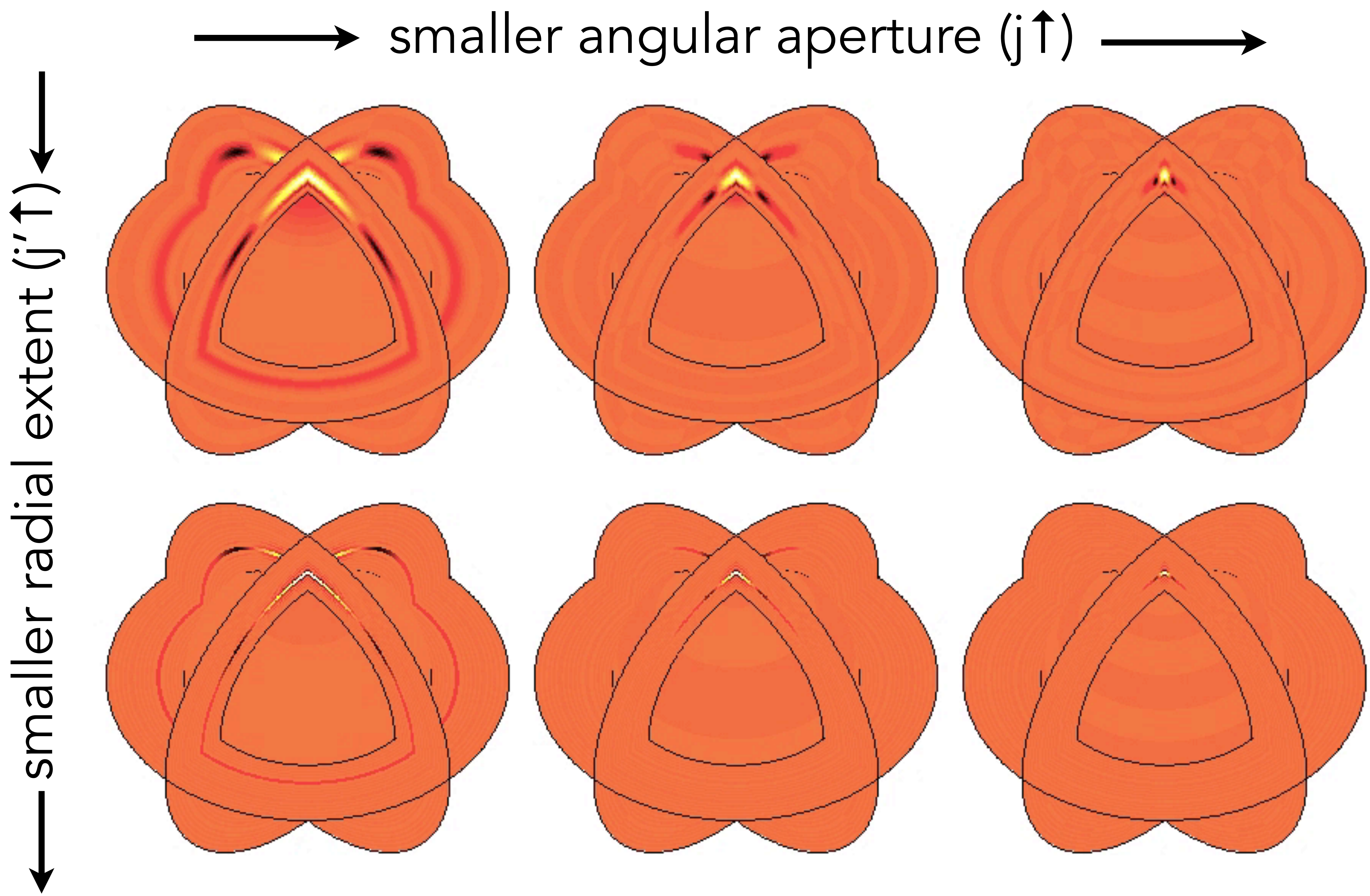
j : angular wavelets

j' : radial wavelets

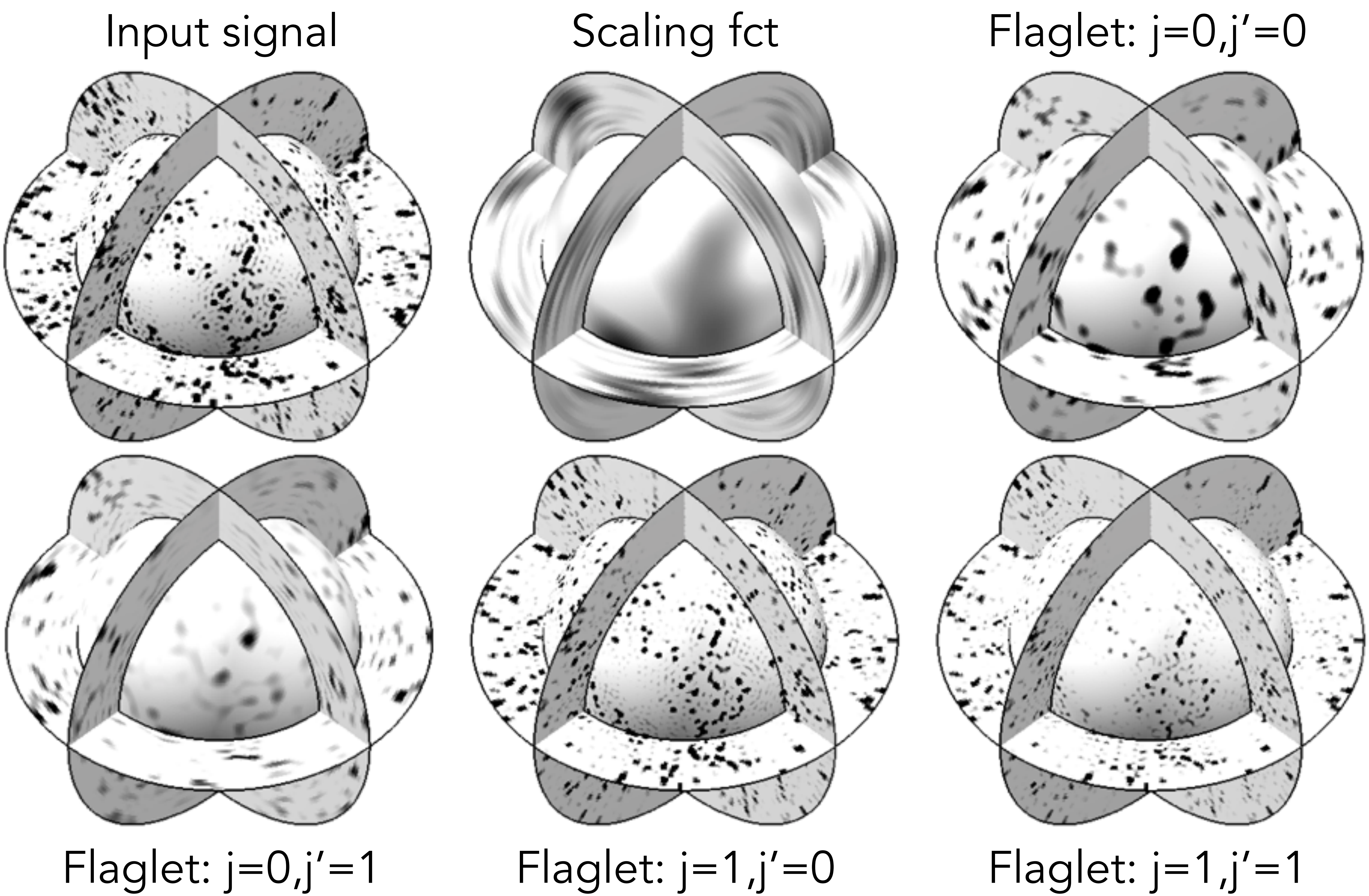
← Flaglets $\Psi^{jj'}$

← Scaling funct Φ

Flaglets $\Psi^{jj'}$

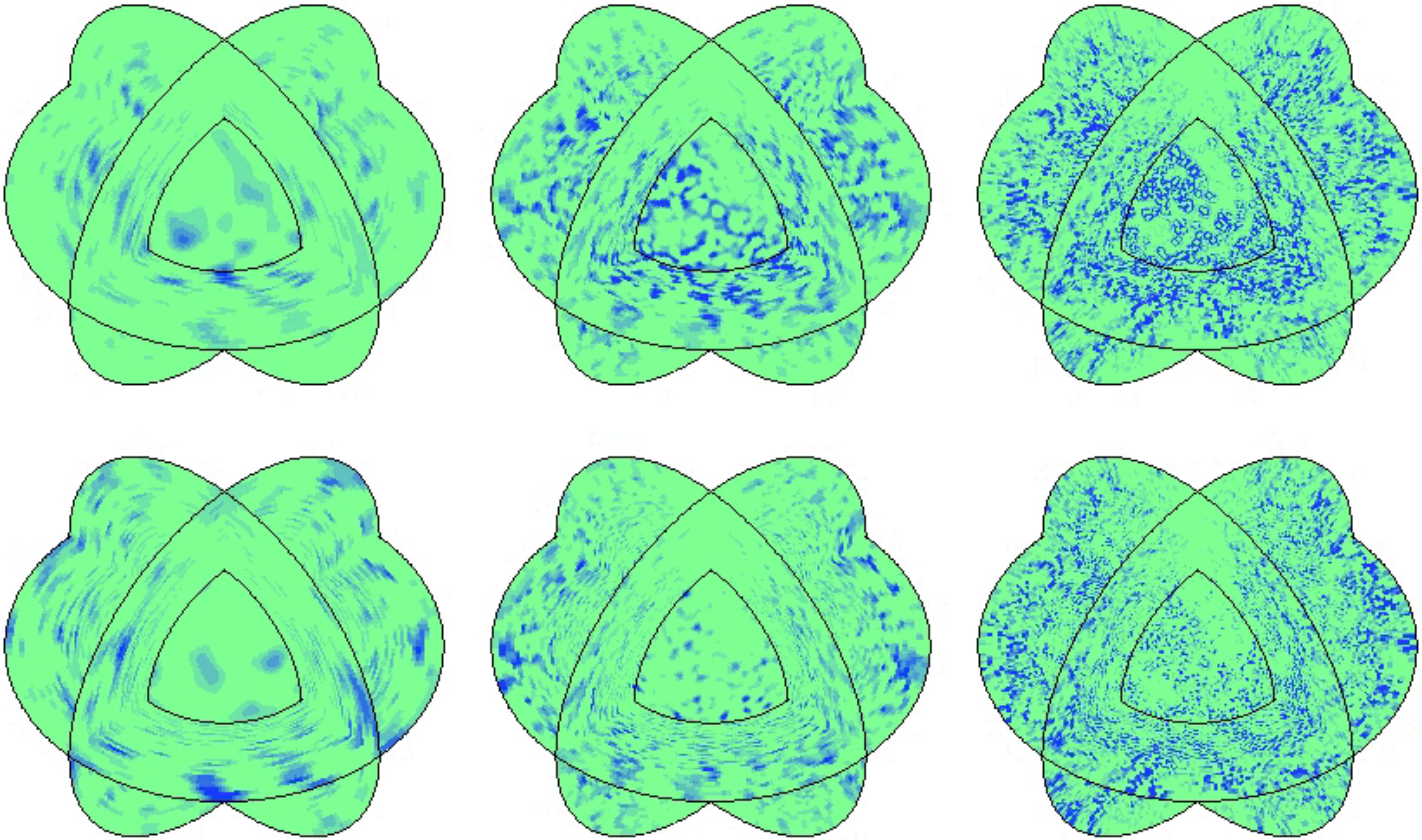


Flaglet transform of Horizon simulation



Underdensities in the Horizon simulation

Ongoing work



Roadmap

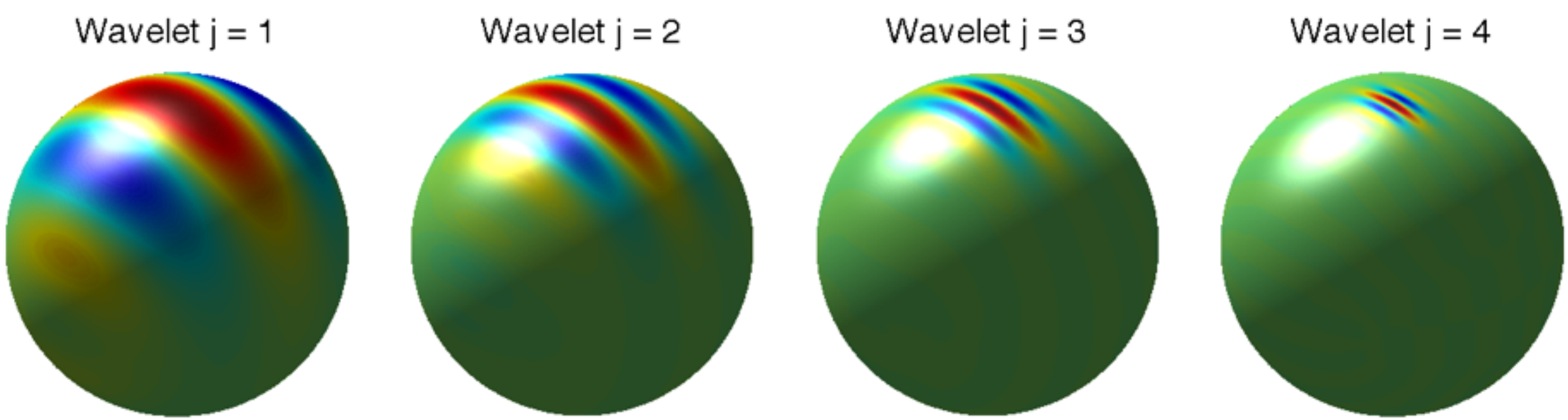
- Galaxy surveys & data on the ball
- Fourier-Laguerre transform on the ball
- Flaglet transform on the ball
- **Spin directional wavelets on the sphere & ball**

Steerable scale-discretised wavelets

- Change $\Psi^j_\ell = \kappa^j(\ell) \quad \Longrightarrow \quad \boxed{\Psi^j_{\ell m} = \kappa^j(\ell) s_{\ell m}}$
- Azimuthal band-limit N

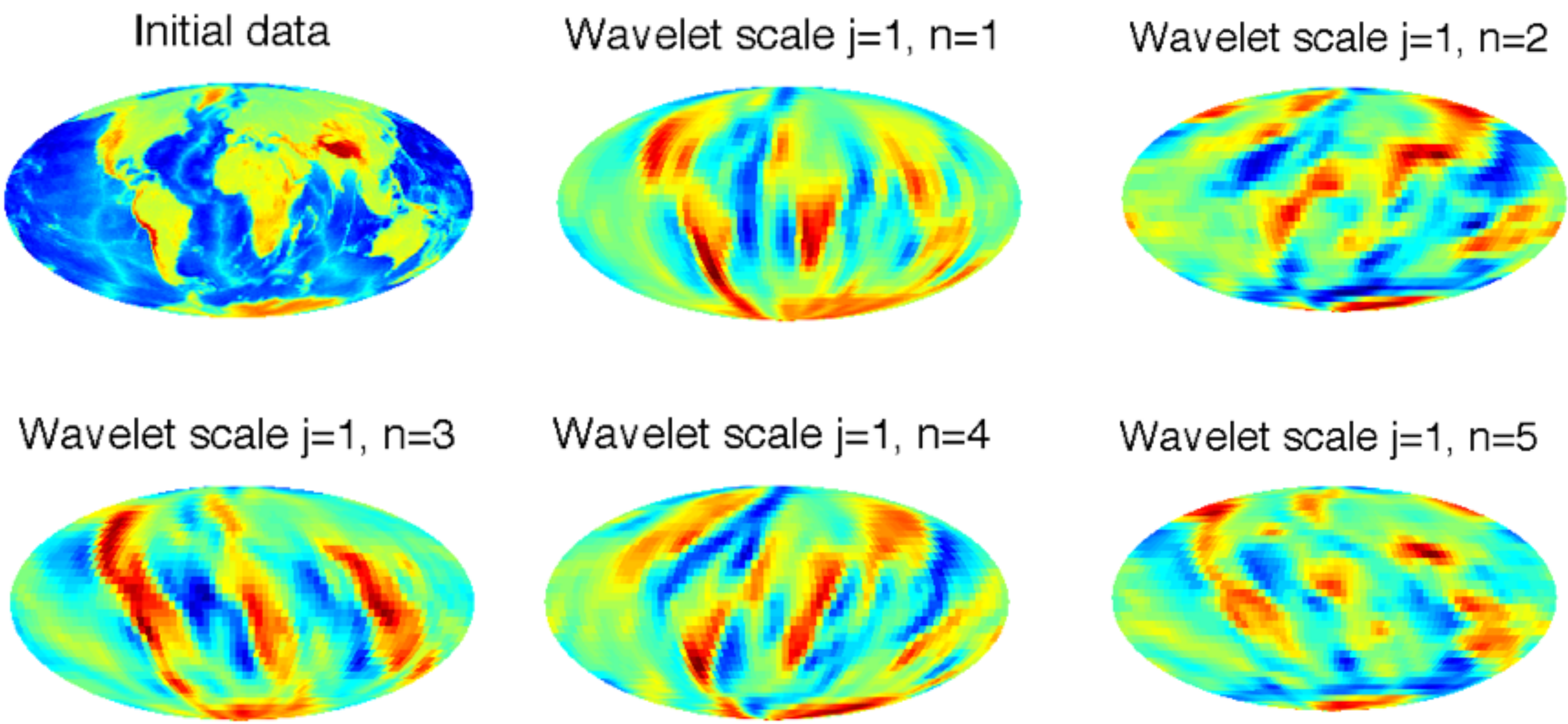
$$s_{\ell m} = 0 \quad \forall \ell, m \text{ with } |m| \geq N$$

- Resulting wavelets:



(Wiaux et al 2009, McEwen et al 2013)

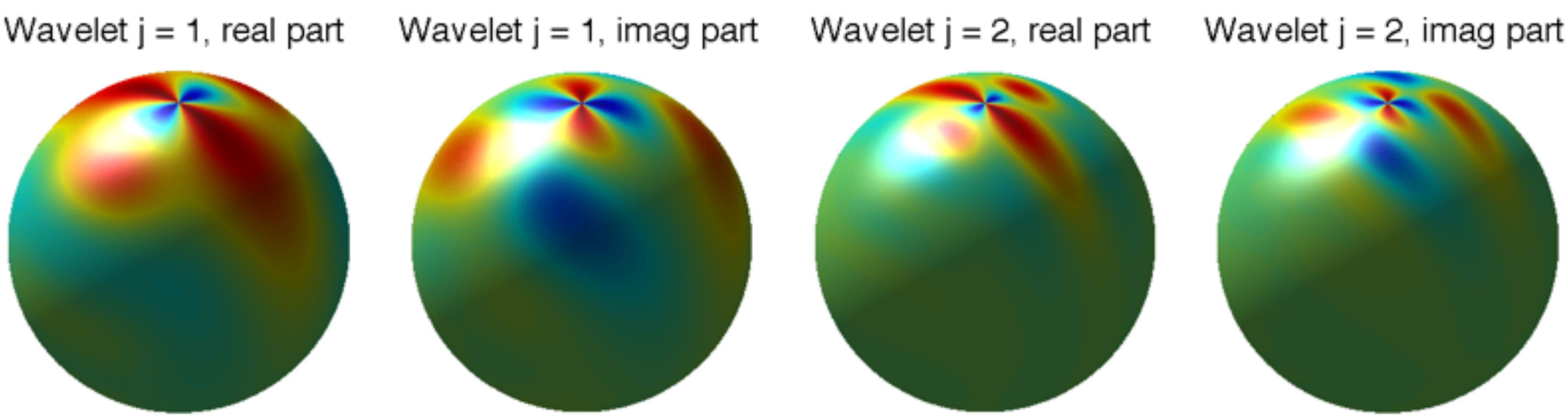
Application to Earth tomography data



- Use SO3 sampling theorem & exact Wigner transform

Spin directional wavelets on the sphere

- Same wavelets tiling but spin harmonic transforms ${}_sY_{\ell m}$



- ***E-B separation through wavelet transform:***

Step 1 : forward **spin** wavelet transform

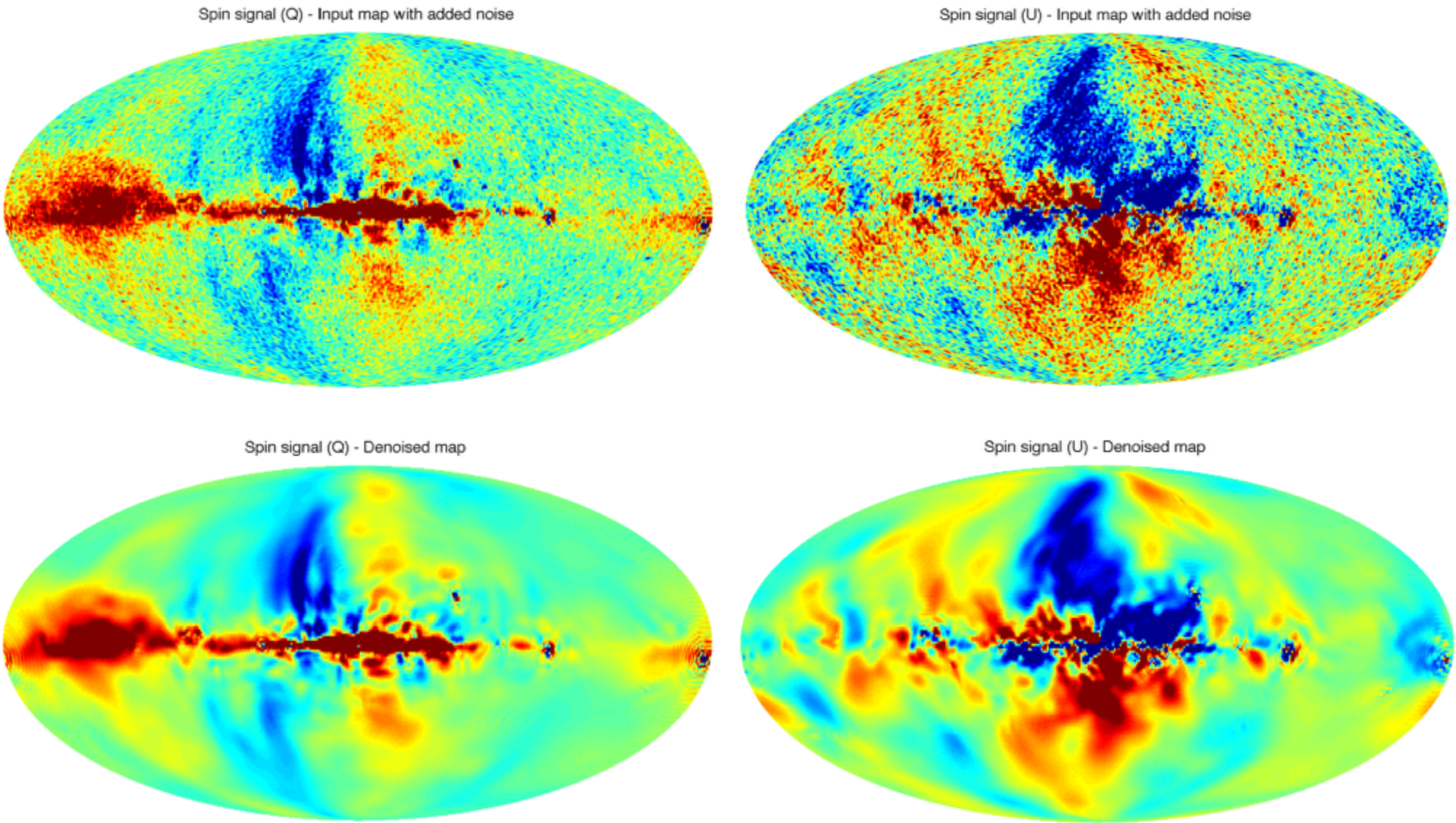
$$(Q + iU)(\theta, \phi) \quad \longrightarrow \quad \{ \ W^{{}_2\Psi_j}(\theta, \phi, \rho) \ \}$$

Step 2 : inverse **scalar** wavelet transform

$$\mathrm{Real}\{W^{{}_2\Psi_j}(\theta, \phi, \rho)\} \quad \longrightarrow \quad -\tilde{E}(\theta, \phi)$$

$$\mathrm{Imag}\{W^{{}_2\Psi_j}(\theta, \phi, \rho)\} \quad \longrightarrow \quad -\tilde{B}(\theta, \phi)$$

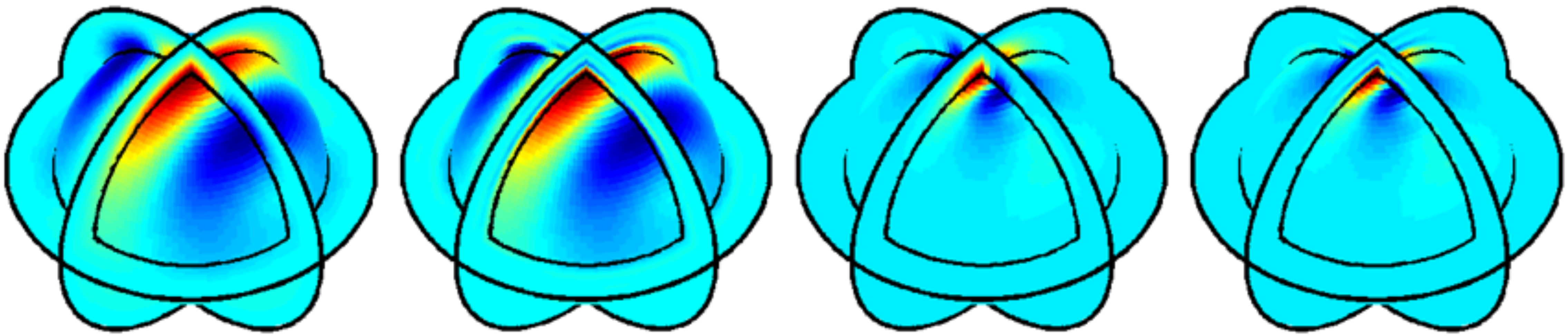
Application: wavelet denoising of spin data



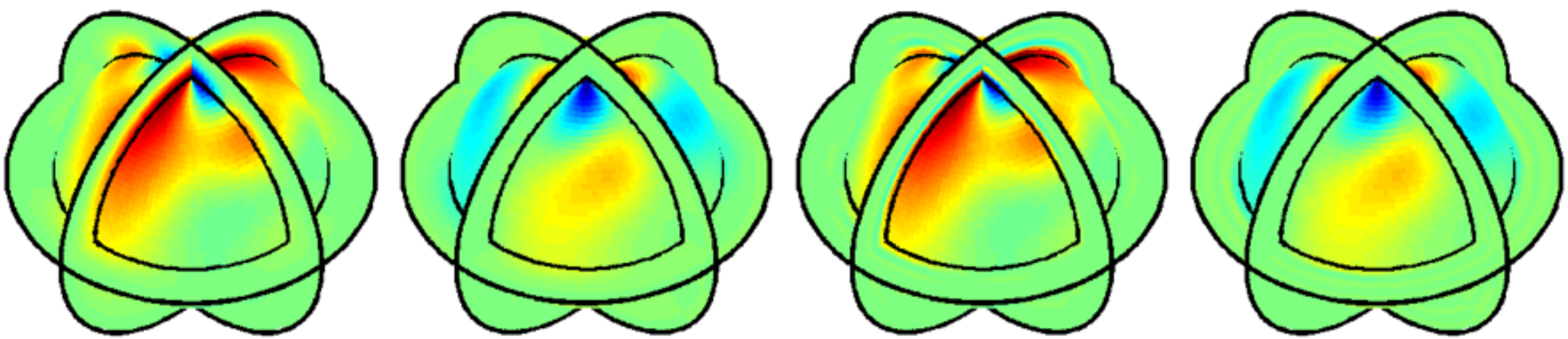
Spin directional **Flaglets**

Ongoing work

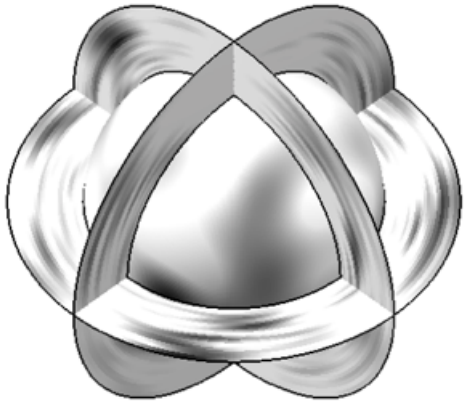
- Directional flaglets



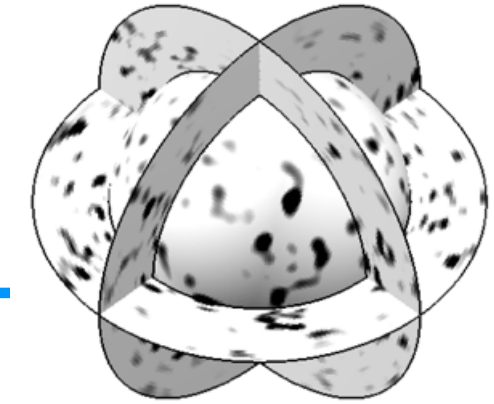
- Spin 2 directional flaglets



- Application to 3D weak lensing analysis (e.g., E-B sep)



Summary & conclusions



- Exact Fourier-Laguerre and Flaglet transforms
- Exact Wigner and spin directional wavelet transforms
- Ongoing: E-B separation, spin directional flaglets
- Future: complex galaxy survey data, cosmic voids, CMB E-B separation, 3D weak lensing, etc

www.flaglets.org *www.s2let.org*

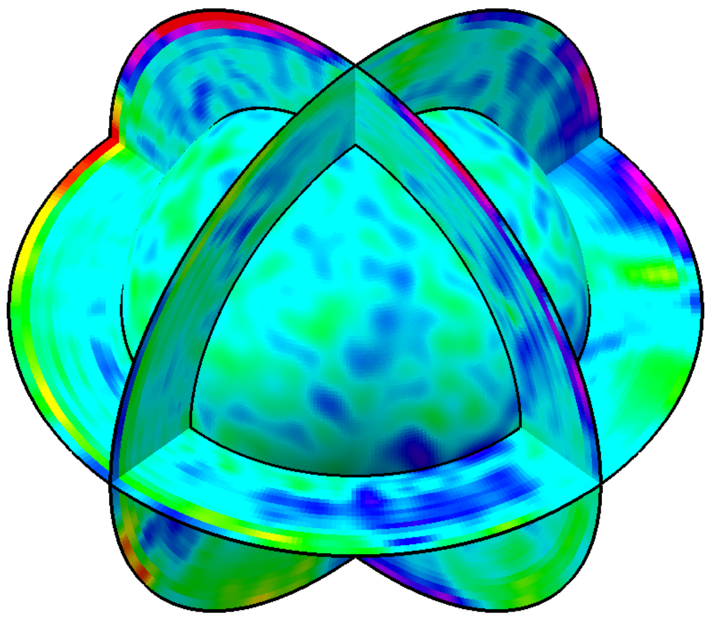
1205.0792, 1308.5480, 1308.5406, + papers in prep

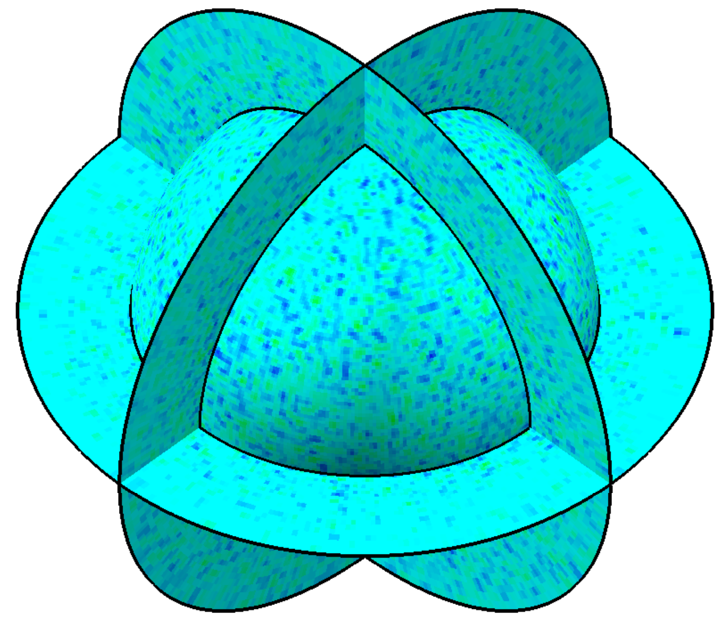
Extra Slides

Flaglet denoising of geophysical model

S40RTS: Ritsema's seismological Earth model of shear wavespeed perturbations in the mantle

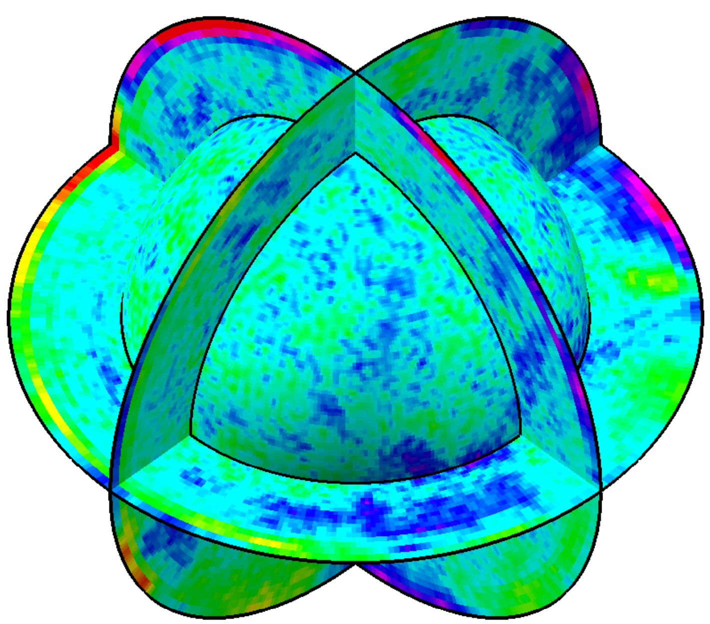
Signal

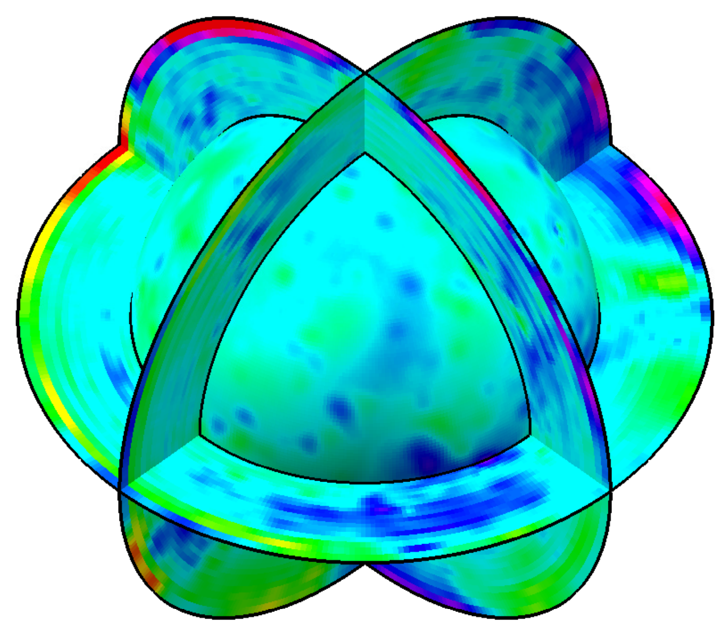




Noise

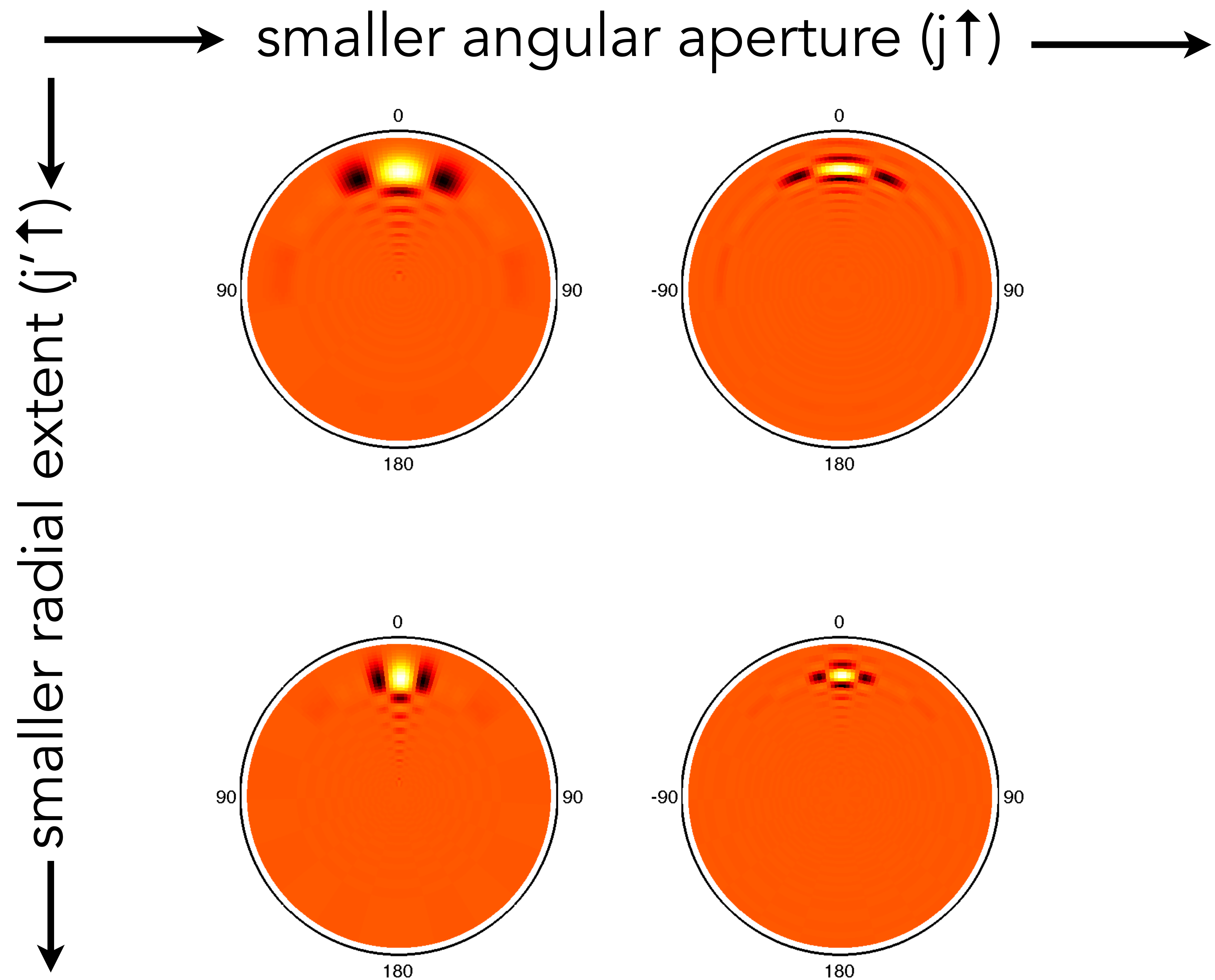
Signal
+noise
SNR=5dB





Denoised
SNR=17

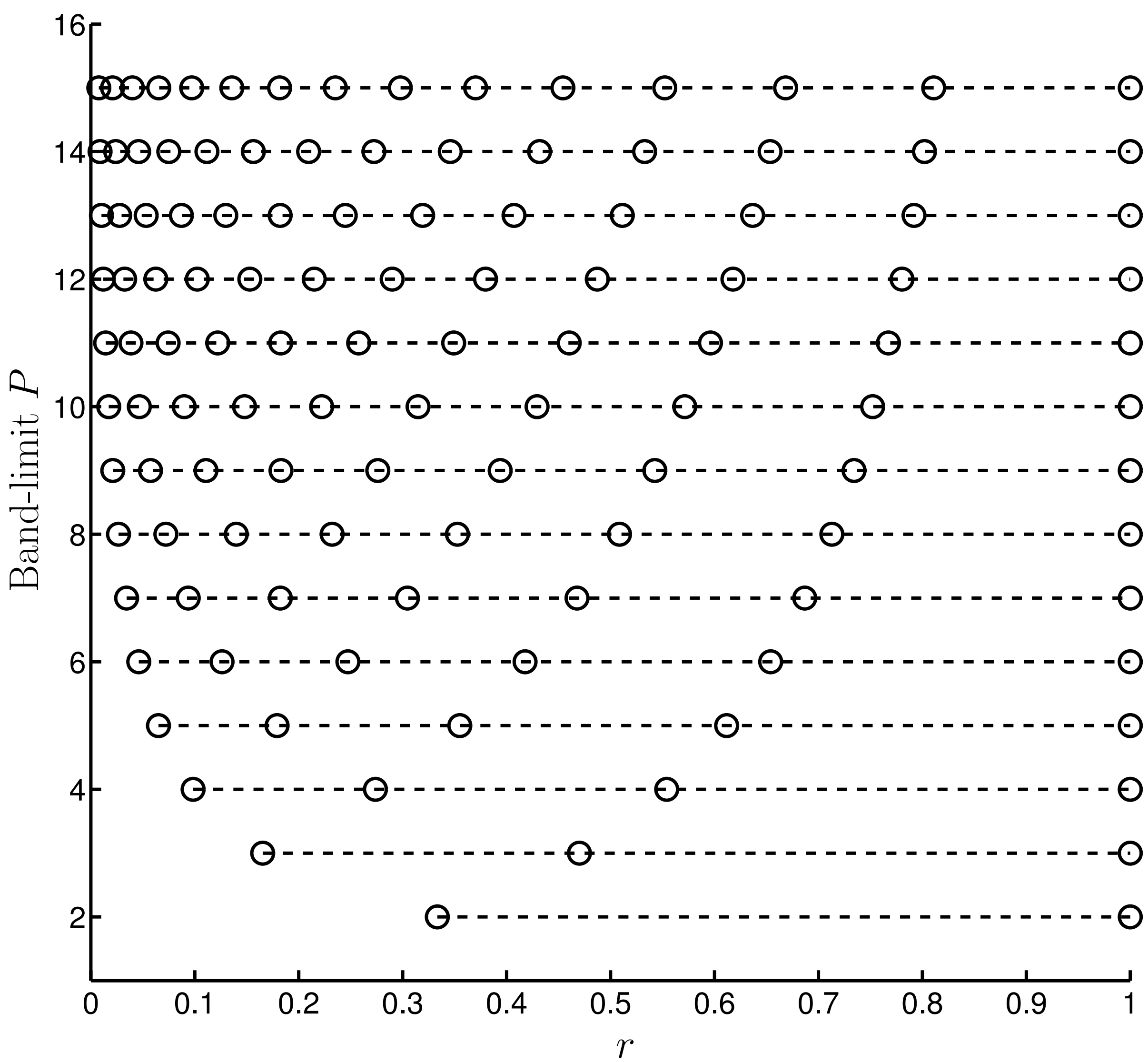
Flaglets $\Psi^{jj'}$



The spherical Laguerre sampling theorem

Spherical Laguerre basis:

$$K_p(r) \equiv \sqrt{\frac{p!}{(p+2)!}} \frac{e^{-r/2\tau}}{\sqrt{\tau^3}} L_p^{(2)}\left(\frac{r}{\tau}\right)$$



- f band-limited at P : projected/ reconstructed on P samples

$$f(r) = \sum_{p=0}^{P-1} f_p K_p(r)$$

$$f_p = \sum_{i=0}^{P-1} w_i f(r_i) K_p(r_i)$$

- Rescaling on any intervals [0,R]
- Sampling denser near origin due to measure $r^2 \mathrm{d}r$

Scale-discretised generating functions (1)

Smooth generating functions

$$s_\lambda(t) \equiv s\left(\frac{2\lambda}{\lambda-1}(t-1/\lambda)-1\right)$$

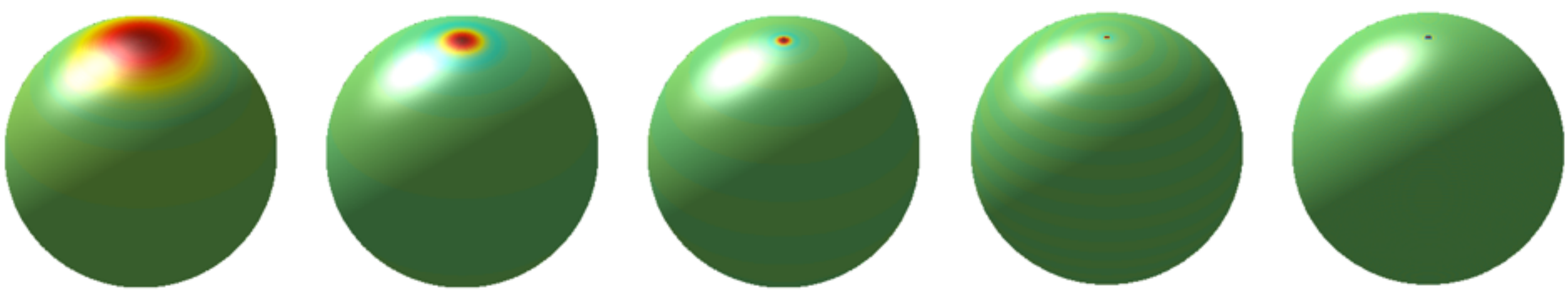
$$s(t) \equiv \begin{cases} e^{-\frac{1}{1-t^2}}, & t \in [-1,1] \\ 0, & t \notin [-1,1] \end{cases}$$

$$k_\lambda(t) \equiv \frac{\int_t^1 \frac{\mathrm{d}t'}{t'} s_\lambda^2(t')}{\int_{1/\lambda}^1 \frac{\mathrm{d}t'}{t'} s_\lambda^2(t')},$$

Azisymmetric wavelets filters

$$\Psi_{\ell m}^j \equiv \sqrt{\frac{2\ell+1}{4\pi}}\,\kappa_\lambda\left(\frac{\ell}{\lambda^j}\right)\delta_{m0}.$$

$$\Phi_{\ell m} \equiv \sqrt{\frac{2\ell+1}{4\pi}}\,\eta_\lambda\left(\frac{\ell}{\lambda^{J_0}}\right)\delta_{m0}.$$



Tilling of Fourier-Laguerre space

$$\kappa_\lambda(t) \equiv \sqrt{k_\lambda(t/\lambda) - k_\lambda(t)} \quad \text{and} \quad \eta_\lambda(t) \equiv \sqrt{k_\lambda(t)}$$

$$\kappa_\nu(t) \equiv \sqrt{k_\nu(t/\nu) - k_\nu(t)} \quad \text{and} \quad \eta_\nu(t) \equiv \sqrt{k_\nu(t)}$$

$$\eta_{\lambda\nu}(t, t') \equiv \sqrt{k_\lambda(t/\lambda)k_\nu(t') + k_\lambda(t)k_\nu(t'/\nu) - k_\lambda(t)k_\nu(t')} \,.$$

Flaglet filters:

$$\Psi^{jj'}_{\ell m p} \equiv \sqrt{\frac{2\ell+1}{4\pi}}\, \kappa_\lambda\left(\frac{\ell}{\lambda^j}\right) \kappa_\nu\left(\frac{p}{\nu^{j'}}\right) \delta_{m0}.$$

$$\Phi_{\ell m p} \equiv \begin{cases} \sqrt{\frac{2\ell+1}{4\pi}}\, \eta_\nu\left(\frac{p}{\nu^{J'_0}}\right) \delta_{m0}, & \text{if } \ell > \lambda^{J_0},\ p \le \nu^{J'_0} \\ \sqrt{\frac{2\ell+1}{4\pi}}\, \eta_\lambda\left(\frac{\ell}{\lambda^{J_0}}\right) \delta_{m0}, & \text{if } \ell \le \lambda^{J_0},\ p > \nu^{J'_0} \\ \sqrt{\frac{2\ell+1}{4\pi}}\, \eta_{\lambda\nu}\left(\frac{\ell}{\lambda^{J_0}}, \frac{p}{\nu^{J'_0}}\right) \delta_{m0}, & \text{if } \ell < \lambda^{J_0},\ p < \nu^{J'_0} \\ 0, & \text{elsewhere.} \end{cases}$$

Fourier-Bessel transform

- Fourier-Bessel : $Z'_{\ell m}(k;\vec{r}) = Y_{\ell m}(\theta,\phi) j_\ell(kr)$
- Eigenfunctions of the Laplacian in 3D spherical coord.

$$
\begin{aligned}
f(\vec{r}) &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}+} \mathrm{d}k k^2 \tilde{f}_{\ell m}(k) Y_{\ell m}(\theta,\phi) j_\ell(kr) \\
\tilde{f}_{\ell m}(k) &= \sqrt{\frac{2}{\pi}} \int_{\mathbb{S}^2} \mathrm{d}\Omega(\theta,\phi) \int_{\mathbb{R}+} \mathrm{d}r r^2 f(r,\theta,\phi) Y^*_{\ell m}(\theta,\phi) j_\ell(kr)
\end{aligned}
$$

- But no sampling theorem for $\int_{\mathbb{R}+} f(r) j_\ell(kr) r^2 \mathrm{d}r$

Connection to Fourier-Bessel analysis (2)

$$\tilde{f}_{\ell m}(k) = \sqrt{\frac{2}{\pi}} \sum_p f_{\ell m p} j_{\ell p}(k) \quad \text{with} \quad j_{\ell p}(k) \equiv \langle K_p | j_\ell \rangle$$

<u>finite sum if band-limited</u> <u>has analytical formula</u>

Details of analytic formula:

$$j_{\ell p}(k) = \sqrt{\frac{p!}{(p+2)!}} \sum_{j=0}^{p} c_j^p \mu_{j+2}^{\ell}(k),$$

$$c_j^p \equiv \frac{(-1)^j}{j!} \binom{p+2}{p-j} = -\frac{p-j+1}{j(j+2)} c_{j-1}^p.$$

$$\mu_j^\ell(k) \equiv \frac{1}{\tau^{j-\frac{1}{2}}} \int_{\mathbb{R}+} \mathrm{d}r r^j j_\ell(kr) e^{-\frac{r}{2\tau}}$$

with

$$\mu_j^\ell(k) = \sqrt{\pi}\ 2^j\ \tilde{k}^\ell\ \tau^{\frac{3}{2}}\ \frac{\Gamma(j+\ell+1)}{\Gamma(\ell+\frac{3}{2})}\ {}_2F_1\left(\frac{j+\ell+1}{2}; \frac{j+\ell}{2}+1; \ell+\frac{3}{2}; -4\tilde{k}^2\right)$$

More details on the Flaglet transform

- Thanks to sampling theorem, flaglet transform easily computed

Projection:

$$W^{\Phi}(\vec{r}) \equiv (f \star \Phi)(\vec{r}) = \langle f | \mathcal{T}_r \mathcal{R}_{(\theta,\phi)} \Phi \rangle$$

$$W^{\Psi^{jj'}}(\vec{r}) \equiv (f \star \Psi^{jj'})(\vec{r}) = \langle f | \mathcal{T}_r \mathcal{R}_{(\theta,\phi)} \Psi^{jj'} \rangle$$

Reconstruction:

$$f(r,\theta,\phi) = \int_{\mathbb{B}^3} W^{\Phi}(\vec{s})(\mathcal{T}_r \mathcal{R}_{(\theta,\phi)} \Phi)(\vec{s}) \mathrm{d}^3\vec{s} + \sum_{jj'} \int_{\mathbb{B}^3} W^{\Psi^{jj'}}(\vec{s})(\mathcal{T}_r \mathcal{R}_{(\theta,\phi)} \Psi^{jj'})(\vec{s}) \mathrm{d}^3\vec{s}$$

- Most efficient: filtering in Fourier-Laguerre space

Projection:

$$W^{\Psi^{jj'}}_{\ell m p} = \sqrt{\frac{4\pi}{2\ell+1}} f_{\ell m p} \Psi^{jj'}_{\ell 0 p} \qquad W^{\Phi}_{\ell m p} = \sqrt{\frac{4\pi}{2\ell+1}} f_{\ell m p} \Phi_{\ell 0 p}$$

Reconstruction:

$$f_{\ell m p} = \sqrt{\frac{4\pi}{2\ell+1}} W^{\Phi}_{\ell m p} \Phi_{\ell 0 p} + \sqrt{\frac{4\pi}{2\ell+1}} \sum_{jj'} W^{\Psi^{jj'}}_{\ell m p} \Psi^{jj'}_{\ell 0 p}$$

E-B separation through wavelet transform

<u>Step 1</u> : forward **spin** wavelet transform

$$(Q + iU)(\theta, \phi) \quad \longrightarrow \quad \{\ W^{{}_2\Psi_j}(\theta, \phi, \rho)\ \}$$

<u>Step 2</u> : inverse **scalar** wavelet transform

$$\mathrm{Real}\{W^{{}_2\Psi_j}(\theta, \phi, \rho)\} \quad \longrightarrow \quad -\tilde{E}(\theta, \phi)$$

$$\mathrm{Imag}\{W^{{}_2\Psi_j}(\theta, \phi, \rho)\} \quad \longrightarrow \quad -\tilde{B}(\theta, \phi)$$