Using the CMB to reconstruct the primordial power spectrum

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Layout

Probing the Primordial Power Spectrum with CMB

- A probe into the primordial universe
- Link between PPS and CMB
- The difficulties of the reconstruction problem

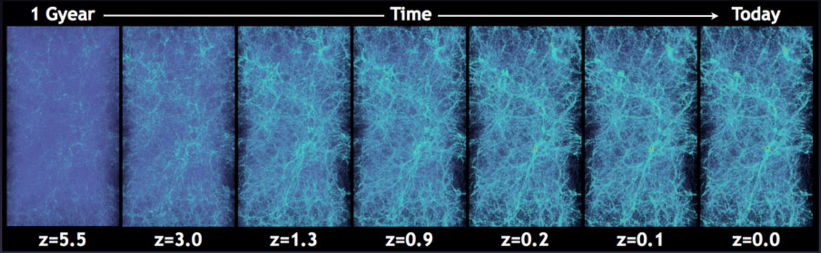
The PRISM algorithm

- The PRISM approach
- Sparse regularisation of the inverse problem
- Variance stabilisation

Results

- WMAP9 simulations and data
- Reconstruction from Planck data
- Test on artificial Planck feature

- Inflation produces initial curvature perturbations
- Large scale structures we see today are seeded by these initial perturbations



Credit: Habib et al./Argonne National Lab

- Different inflation models predict different power spectra for these initial perturbations

- The simplest models of inflation predict a near-scale invariant power spectrum:

$$P(k) = A_s \left(\frac{k}{k_p} \right)^{n_s - 1}$$

- More complex models can produce features on the primordial power spectrum:

$\Longrightarrow$ Having access to the PPS would allow us to discriminate between different models.

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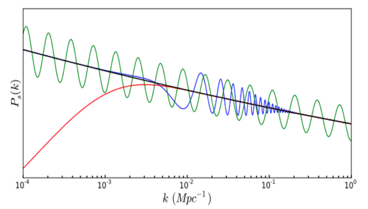

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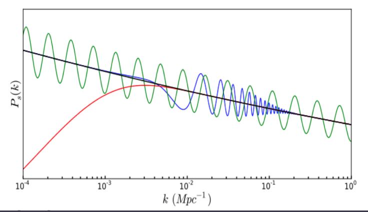

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How to probe **today** the initial perturbations ?

- A number of complementary probes:
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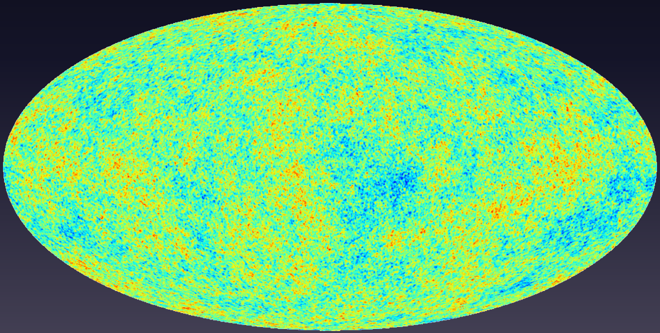

LGMCA Joint reconstruction from Planck PR1 and WMAP 9-year, CEA-irfu, http://www.cosmostat.org/planck_PR1.png

How are the CMB anisotropies linked to the Primordial Power Spectrum ?

- The CMB map is analysed in terms of its spherical harmonics power spectrum C_ℓ^{th}:

$$\langle a_{\ell m} a^*_{\ell' m'} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_\ell^{\mathrm{th}}$$

- The power spectrum of the CMB can be obtained by convolution of the PPS:

$$C_\ell^{\mathrm{th}} = 4\pi \int_0^\infty \mathrm{d}\ln k \Delta_\ell^2(k) P(k)$$

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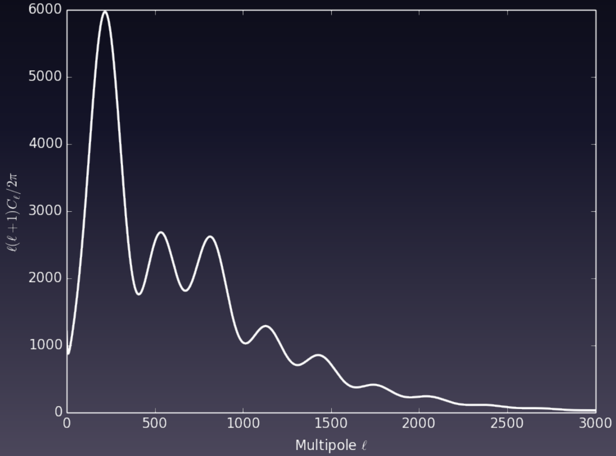
The linear inverse problem to solve

$$C_\ell^{\mathrm{th}} = \mathbf{T}P(k)$$



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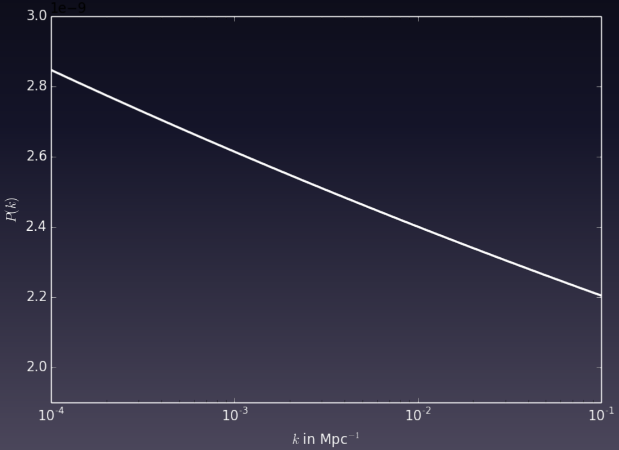
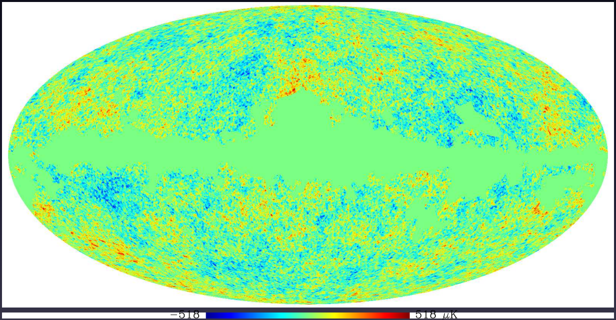


- The angular power spectrum C_ℓ^{th} is unknown. We only have access to the **pseudo-power spectrum**, measured from a masked, noisy map.



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$$\widetilde{C}_\ell = \frac{1}{2\ell+1}\sum_{m=-\ell}^{\ell} |\tilde{a}_{\ell m} + \tilde{n}_{\ell m}|^2$$

  with

$$\langle \widetilde{C}_\ell \rangle \equiv C_\ell = \sum_{\ell'} M_{\ell\ell'} C_{\ell'}^{\rm th} + N_\ell$$

  where $M_{\ell\ell'}$ is the Master matrix (Hivon et al. (2002)).

- Due to sample variance, $\widetilde{C}_\ell$ is contaminated by a multiplicative noise Z_ℓ:

$$\widetilde{C}_\ell = C_\ell^{\rm th} Z_\ell$$

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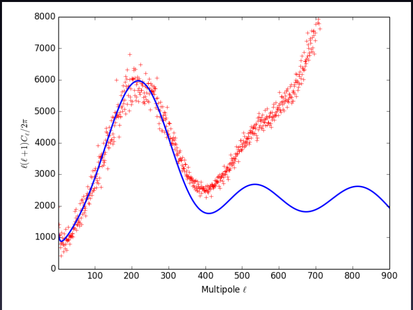
Complete formulation of the problem:

$$\widetilde{C}_\ell = \left(\mathbf{M}\mathbf{T}P(k) + N_\ell\right) Z_\ell$$



- The angular transfer function $\mathbf{T}$ is not invertible
  $\Longrightarrow$ Ill-posed linear inverse problem: no unique, stable solution
- The multiplicative noise Z_ℓ is difficult to handle

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- Aims to reconstruct a non parametric PPS, in particular localised features
- Based on sparse regularisation of ill-posed inverse problems
- Uses a wavelet based analysis sparsity prior (multiscale approach)
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Considering a general linear problem of the form:

$$Y = \mathbf{A}X_0 + N$$

An approximation of X_0 can be recovered by imposing a sparsity promoting penalty on the solution in a dictionary Φ.

$$\boxed{\min_X \frac{1}{2} \parallel Y - \mathbf{A}X \parallel_2^2 + \lambda \parallel \Phi^t X \parallel_1}$$

Simple example: **Deblurring**

- In our case, in the absence of noise, the PPS reconstruction problem could be written has:

$$\min_X \frac{1}{2} \parallel C_\ell - (\mathbf{MT}X + N_\ell) \parallel_2^2 + \lambda \parallel \mathbf{\Phi}^t X \parallel_1$$

with $\mathbf{\Phi}$ a wavelet dictionary.

- This problem can be solved using the Fast Iterative Soft Thresholding Algorithm (Beck & Teboulle (2009)).

$\Longrightarrow$ We can keep this approach if we can build an estimator of $R(X) = C_\ell - (\mathbf{MT}X + N_\ell)$ contaminated by additive Gaussian noise

- The variance stabilisation used in PRISM is derived from the Wahba VST (c.f. TOUSI paper, Paykari et al. (2012)):

$$\mathcal{T} : x \in \mathbb{R}^+ \mapsto \frac{\ln x - \mu_L}{\sigma_L} \quad \Longrightarrow \quad \mathcal{T}(\widetilde{C}_\ell) = \frac{\ln C_\ell}{\sigma_L} + \epsilon_\ell$$

with $\epsilon_\ell \sim \mathcal{N}(0,1)$. $\mu_L = \psi_0(L/2) - \ln(L/2)$ and $\sigma_L^2 = \psi_1(L/2)$, where ψ_m is the polygamma function $\psi_m(t) = \frac{d^{m+1}}{dt^{m+1}} \ln \Gamma(t)$.

- Using the VST, and linearising the logarithm for small residuals $R(X)$:

$$\overline{R}(X) = R_\ell(X) + n_\ell$$

where n_ℓ can be assumed to be additive and Gaussian.

Some practical considerations:

- We adjust the sparsity constraint based on the level of noise.
  $\Longrightarrow$ We have one global regularisation parameter expressed as a $k\sigma$ level.
- We use a reweighted ℓ_1 algorithm.
  $\Longrightarrow$ Reduces biases on the solution

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Goal of this study

Demonstrate that without parameter tweaking PRISM is able to reconstruct 3 types of PPS on WMAP9 simulations:

- near scale invariant: $n_s = 0.972$
- PPS with a running: $n_s = 0.972$, $\alpha_s = -0.017$
- near scale invariant + localised feature around $k = 0.03 \mathrm{Mpc}^{-1}$

Details of the simulations:

- 2000 independent CMB + noise realisations for each type of PPS at the WMAP channel level
- Channel maps processed through the LGMCA pipeline to yield one noisy map per realisation at 15 arcmin resolution
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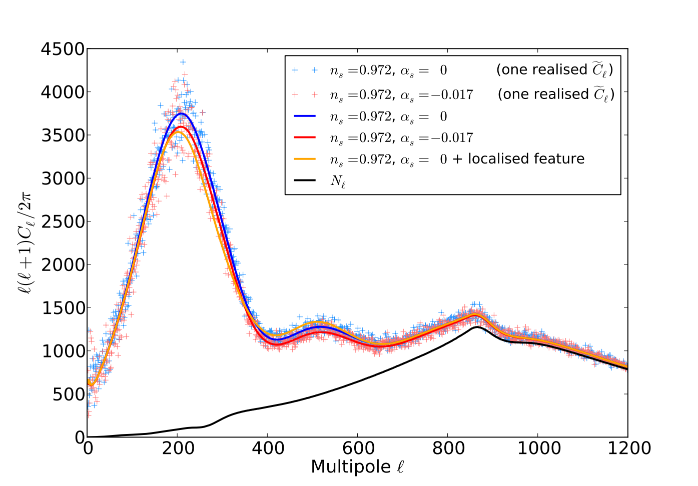
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n_s =0.972, α_s = 0
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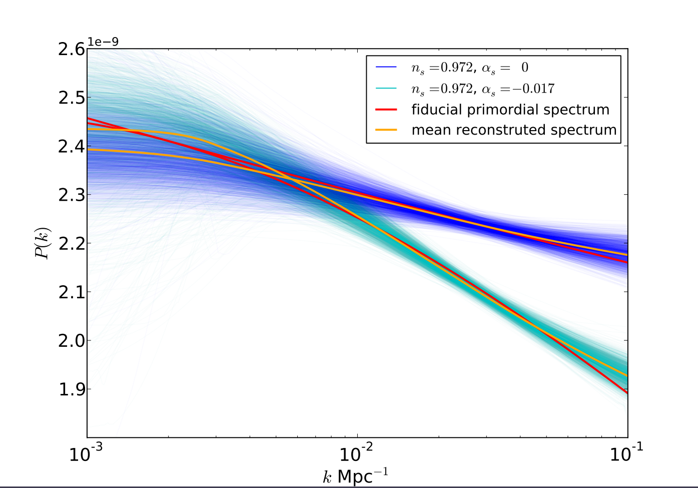
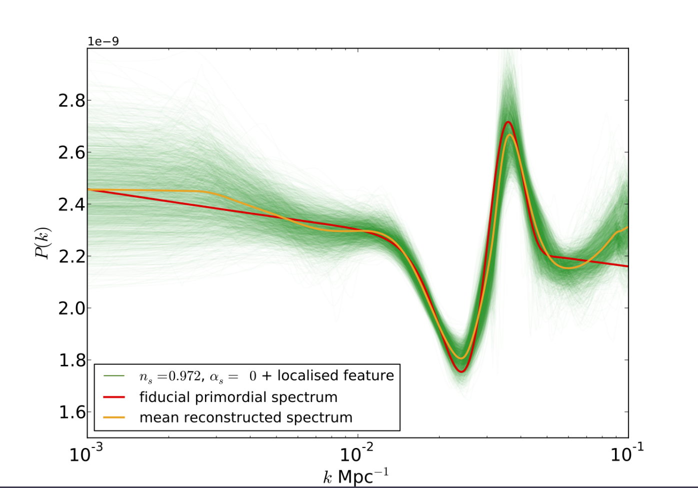
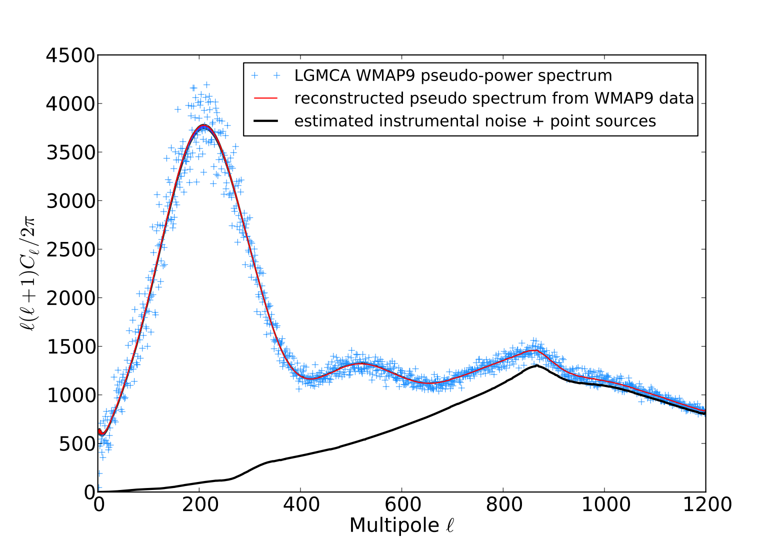
n_s =0.972, α_s = 0 + localised feature
N_ℓ
$\ell(\ell+1)C_\ell/2\pi$
4500
4000
3500
3000
2500
2000
1500
1000
500
0
0
200
400
600
800
1000
1200
Multipole ℓ


1e−9
2.6
2.5
2.4
2.3
2.2
2.1
2.0
1.9
$P(k)$
10^{-3}
10^{-2}
10^{-1}
k Mpc^{-1}
n_s =0.972, α_s = 0
n_s =0.972, α_s =−0.017
fiducial primordial spectrum
mean reconstruted spectrum


1e−9
2.8
2.6
2.4
2.2
2.0
1.8
1.6
$P(k)$
n_s =0.972, α_s = 0 + localised feature
fiducial primordial spectrum
mean reconstructed spectrum
10^{-3}
10^{-2}
10^{-1}
k Mpc^{-1}

Reconstruction from WMAP9 data:



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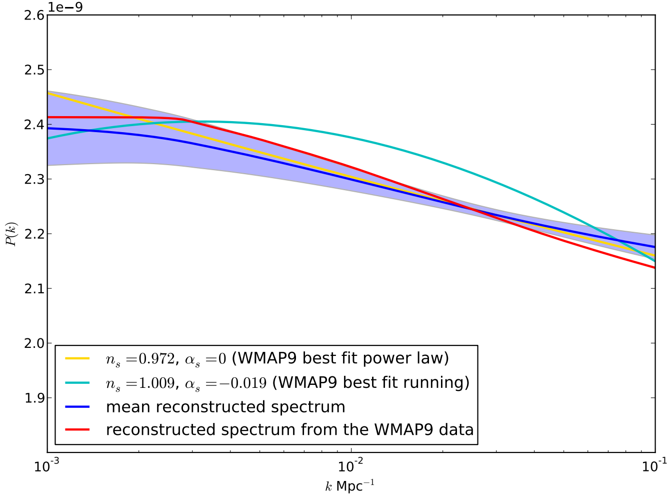
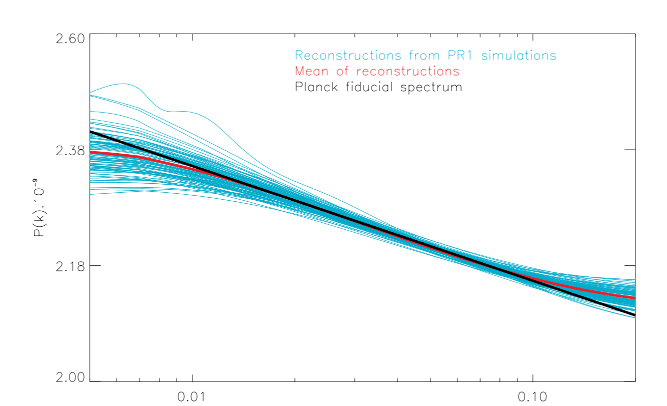
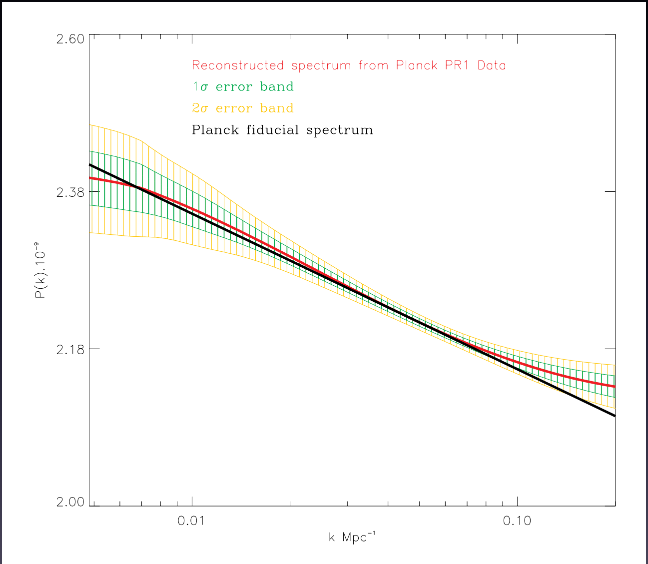


Reconstruction from Planck data:

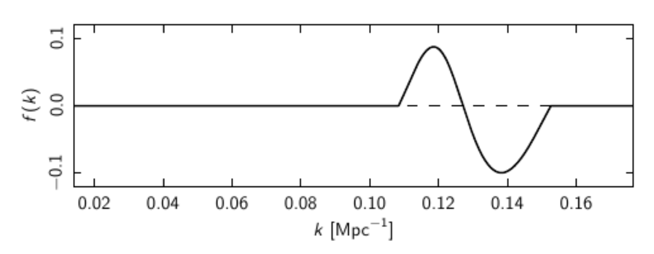
- We use the LGMCA Planck PR1 power spectrum (Bobin et al. (2014))
- We produce 100 CMB realisations from the same pipeline with the best fit near scale invariant power spectrum

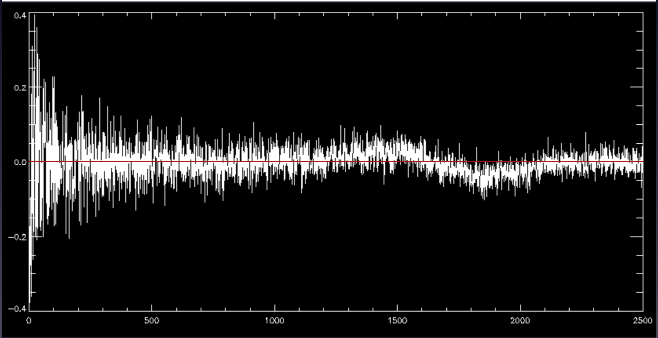


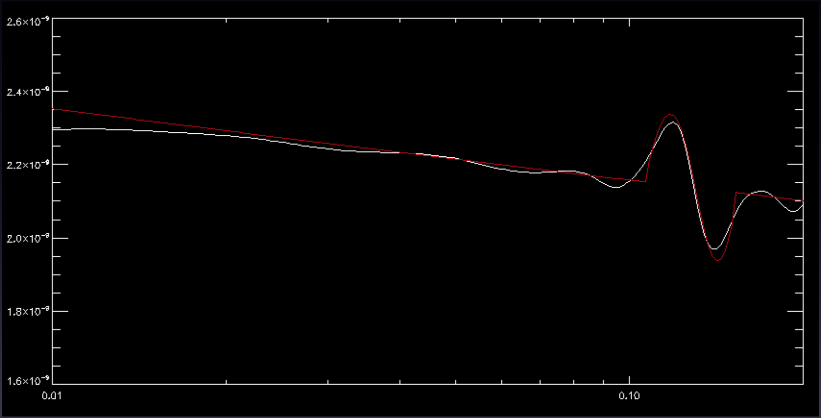

Reconstructed spectrum from Planck PR1 Data
1σ error band
2σ error band
Planck fiducial spectrum
2.60
2.38
2.18
2.00
P(k).10⁻⁹
0.01
0.10
k Mpc⁻¹

Conclusions from simulations and data

- On simulations we recover the shapes of the test power spectra
- We do not detect significant deviations from the best fit near scale invariant power spectrum


0.1
0.0
−0.1
$f(k)$
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
k [Mpc^{-1}]


0.4
0.2
0.0
−0.2
−0.4
0
500
1000
1500
2000
2500


2.6×10⁻⁹
2.4×10⁻⁹
2.2×10⁻⁹
2.0×10⁻⁹
1.8×10⁻⁹
1.6×10⁻⁹
0.01
0.10

Conclusion

- Sparsity based **non-parametric method** to recover primordial power spectrum from masked noisy CMB power spectra
- Multiscale approach, sensitive to both large and local features
- Not detected strong deviations from WMAP9 or Planck PR1 near-scale invariant fiducial primordial power spectra

Reproducible research:

The codes are available at
http://www.cosmostat.org/isap.html
http://www.cosmostat.org/prism.html