Slepian functions on the sphere:

Applications in cosmology and geophysics

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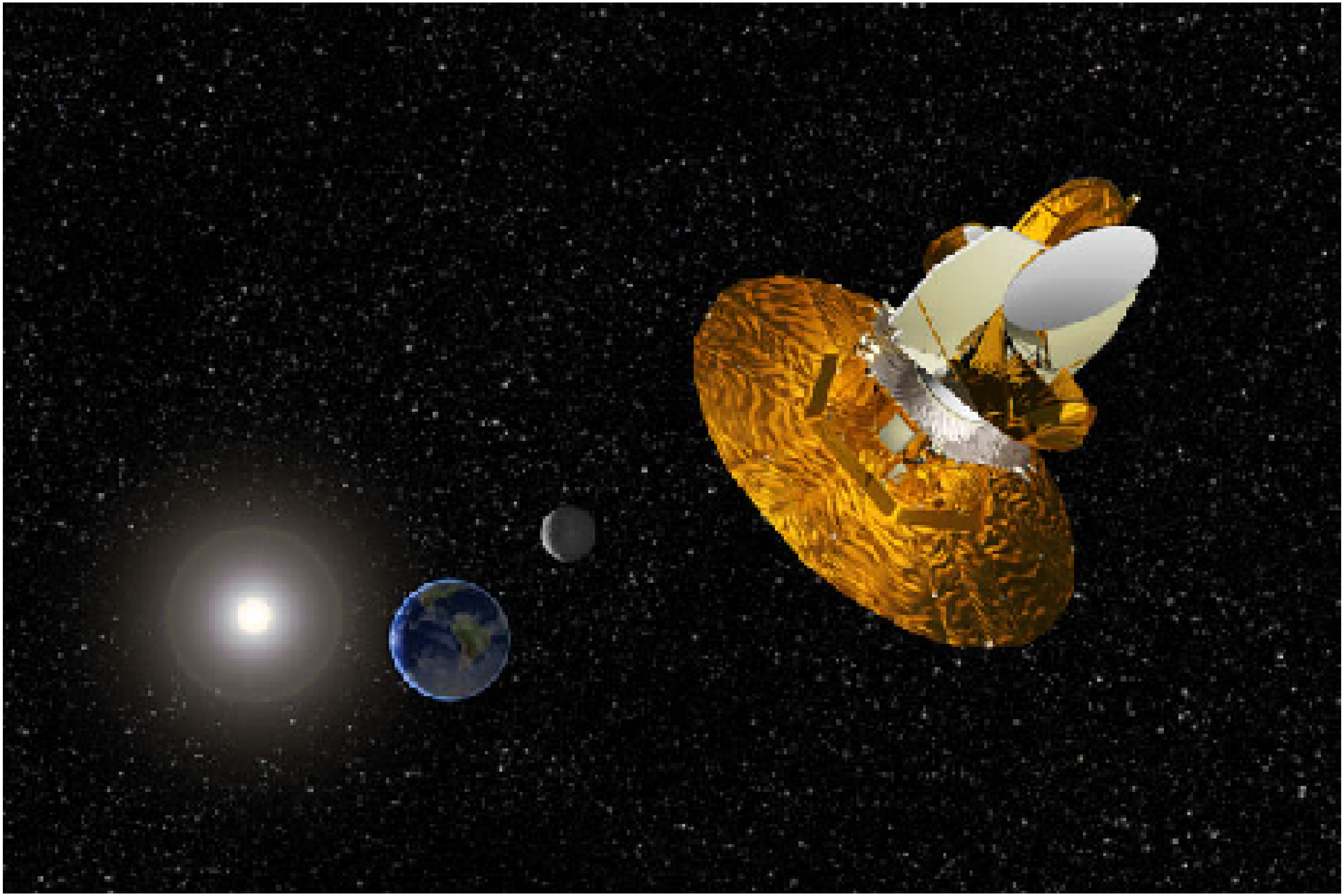
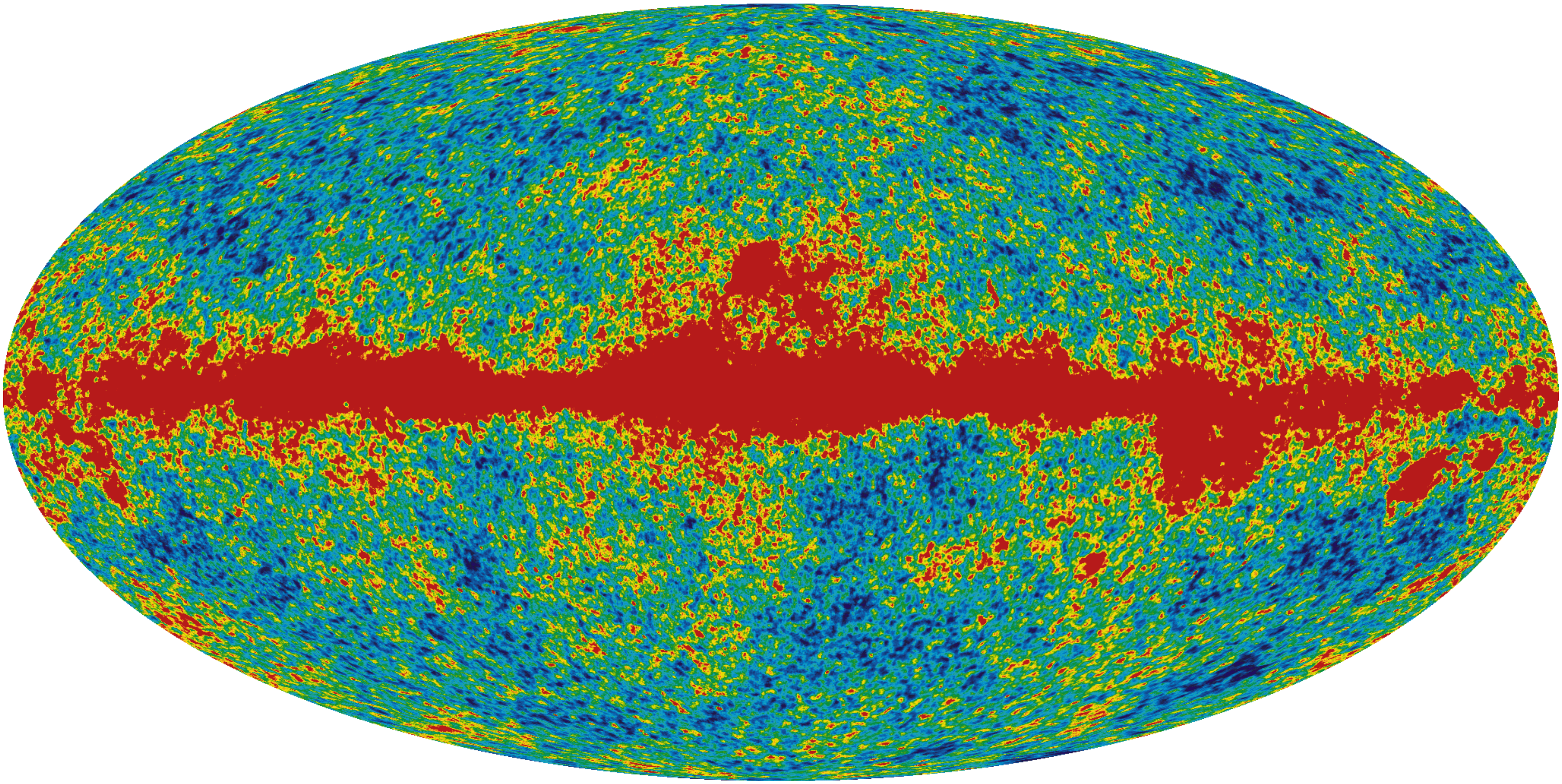
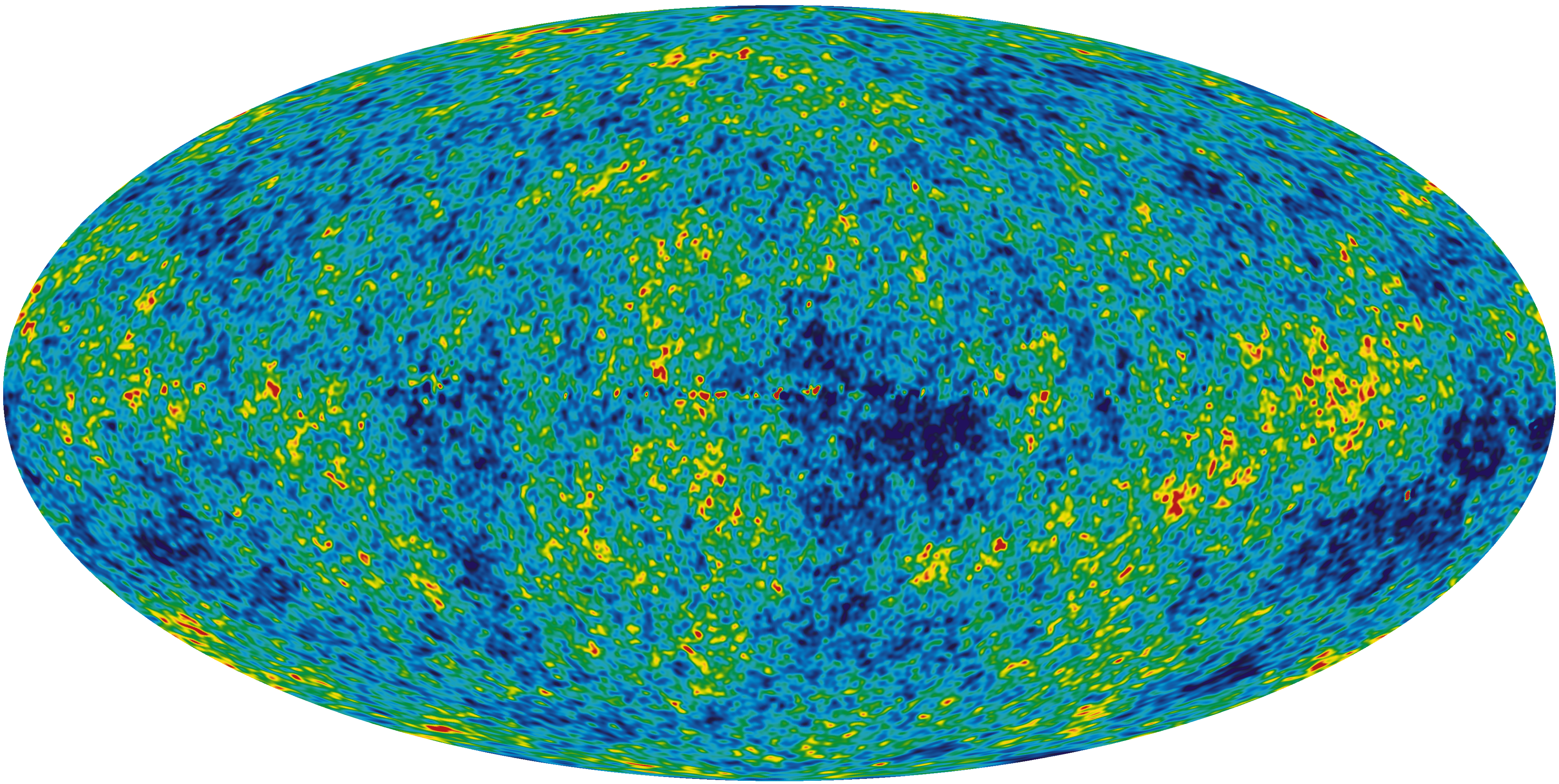
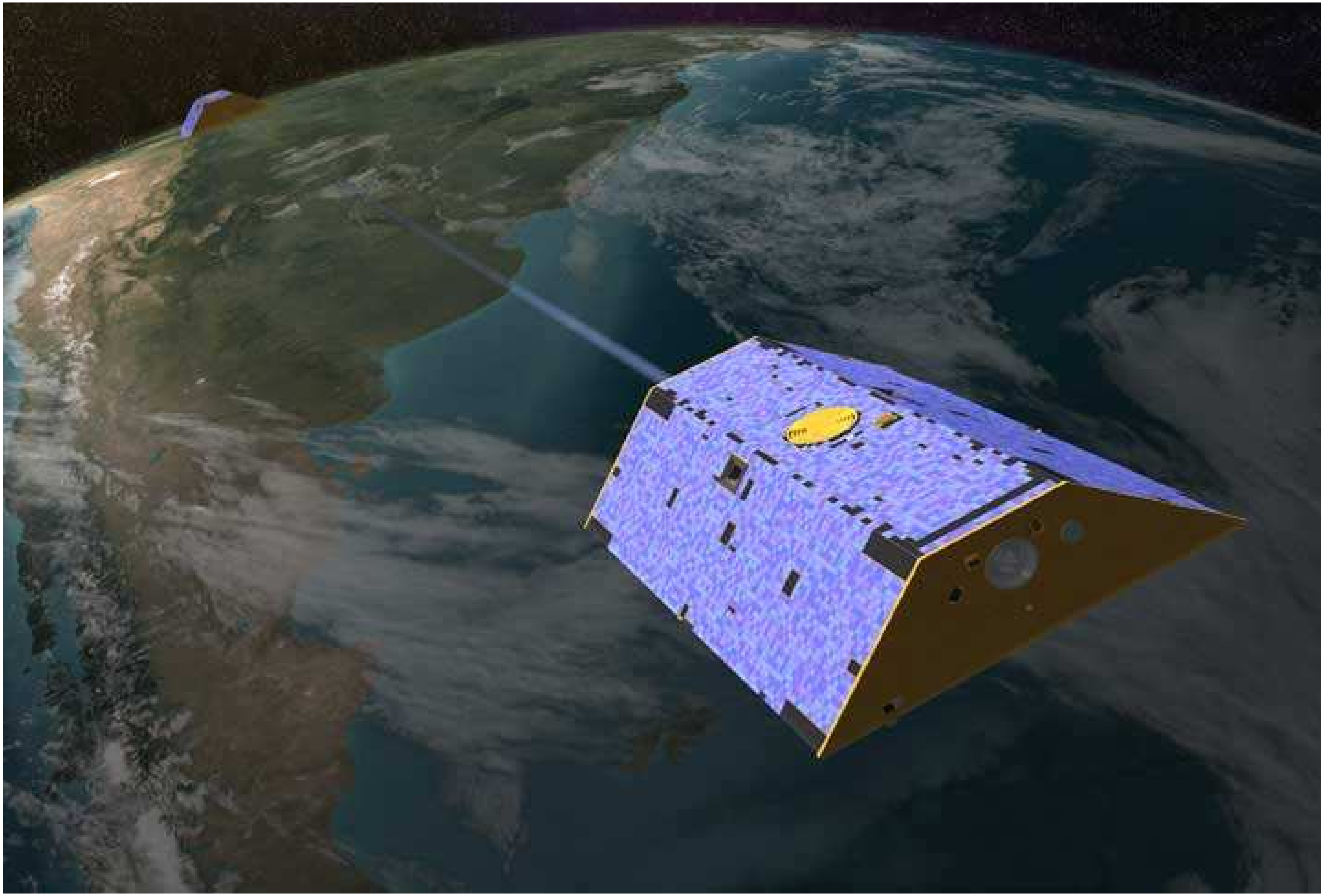


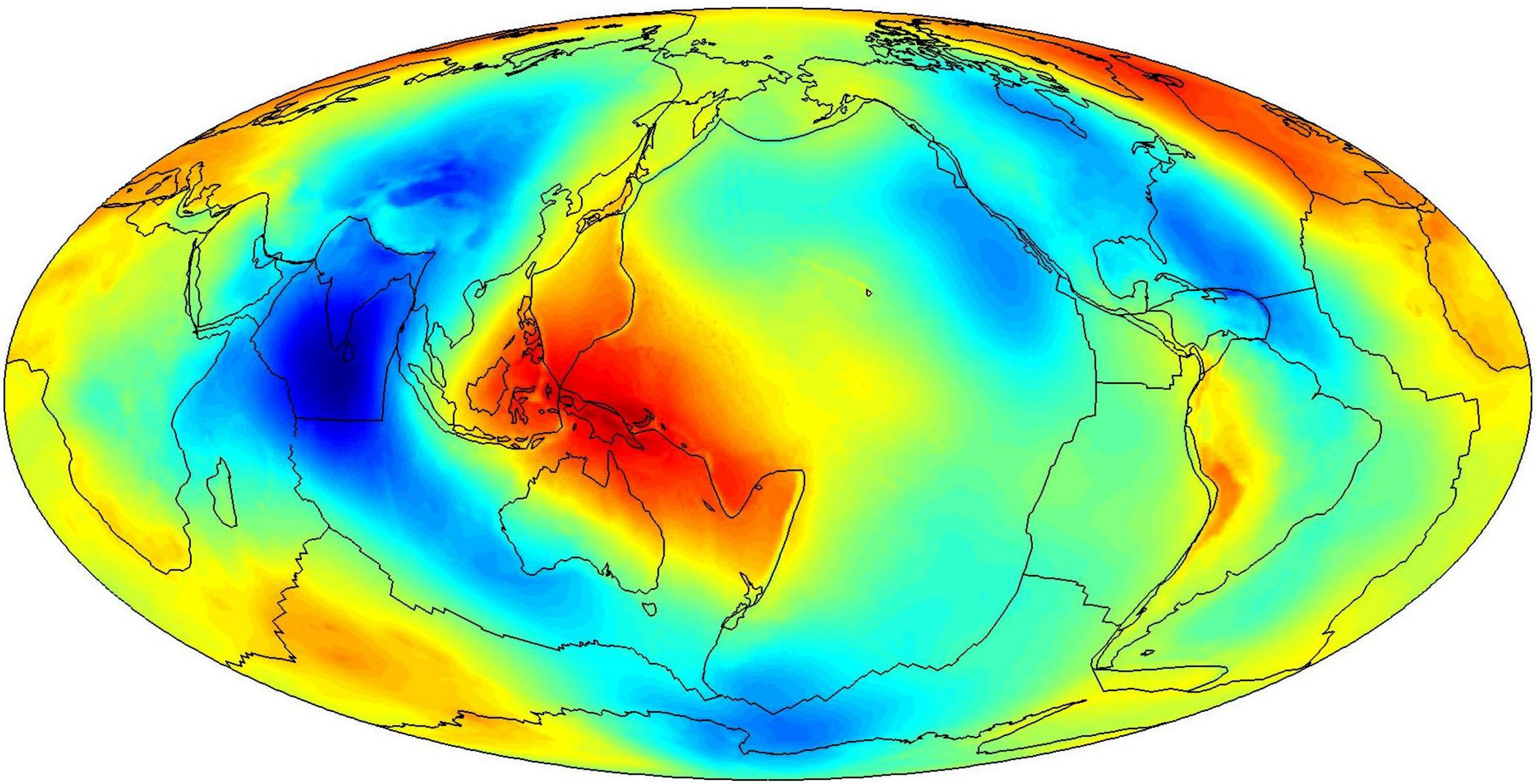
WMAP & Cosmic Background Radiation — 2

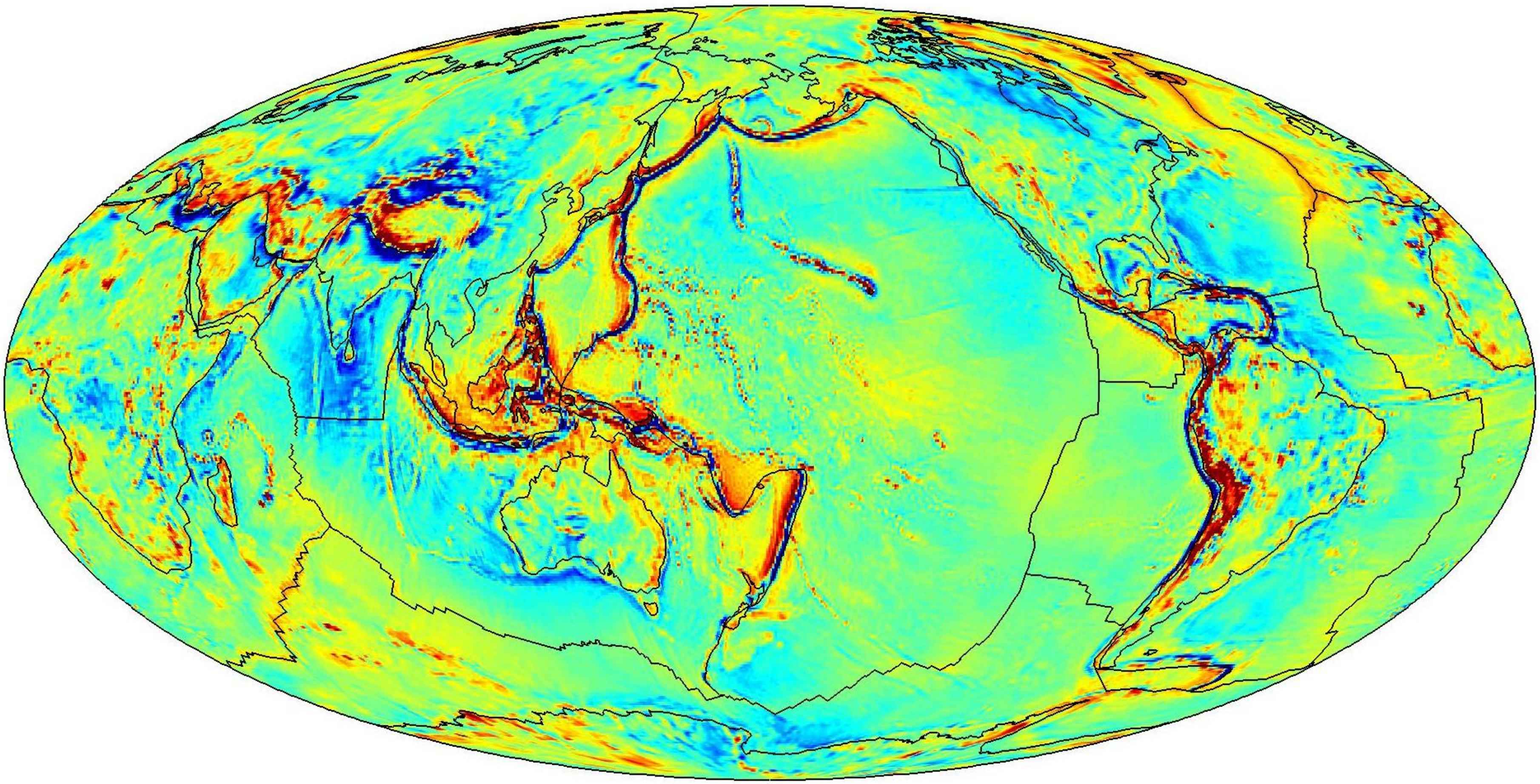




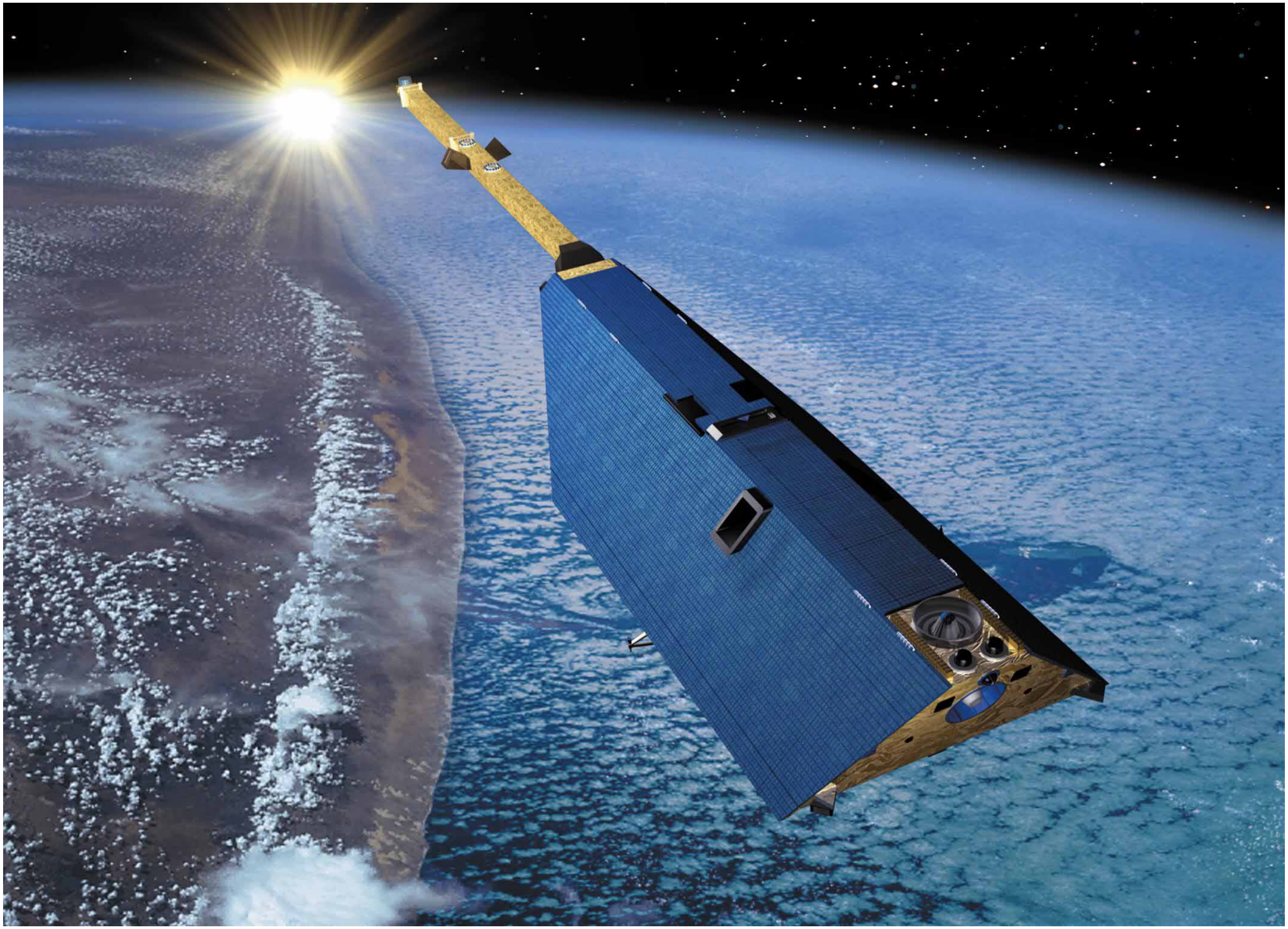
GRACE & Earth's Gravity Field — 1



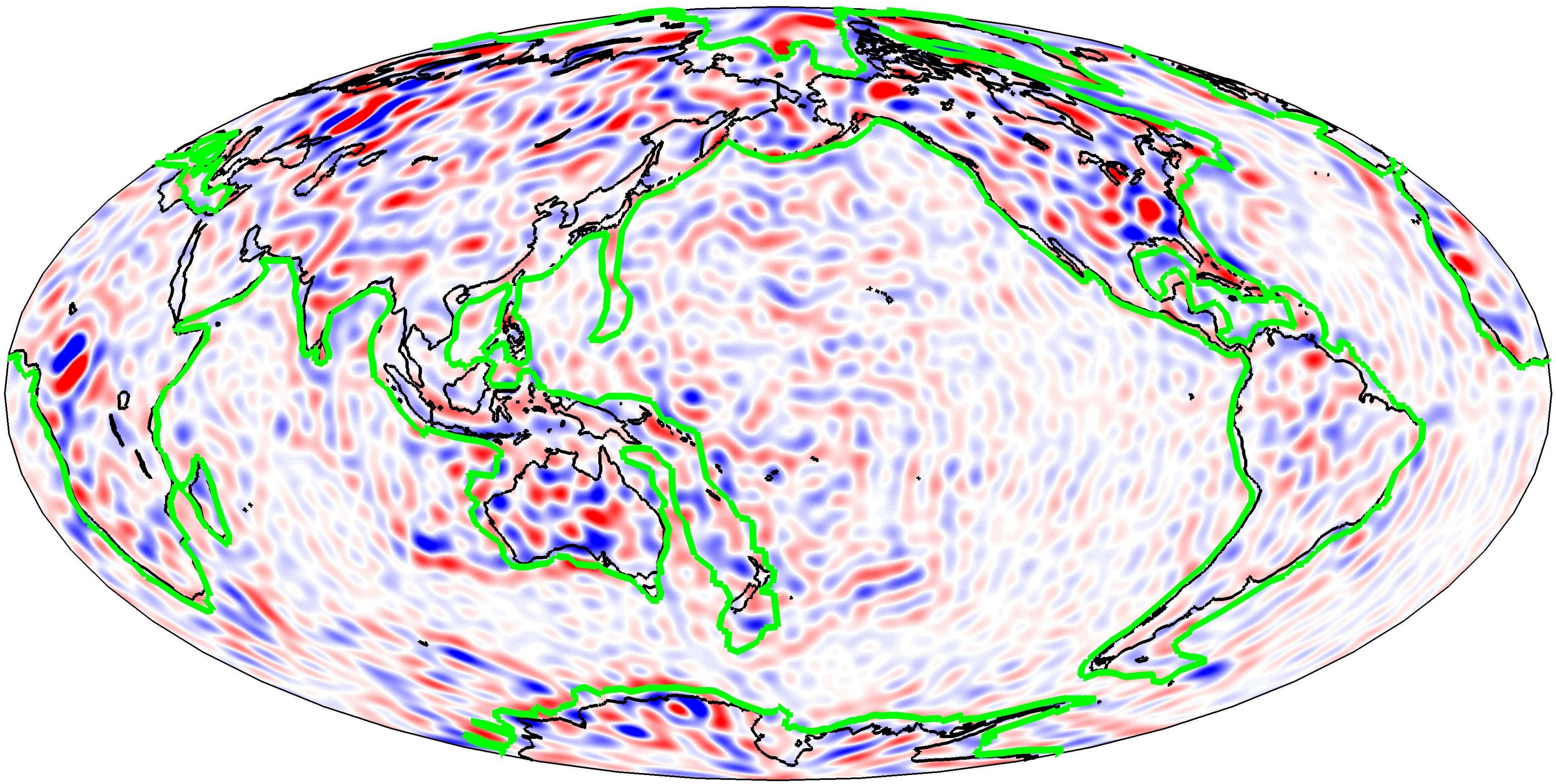




CHAMP & Earth's Magnetic Field — 1

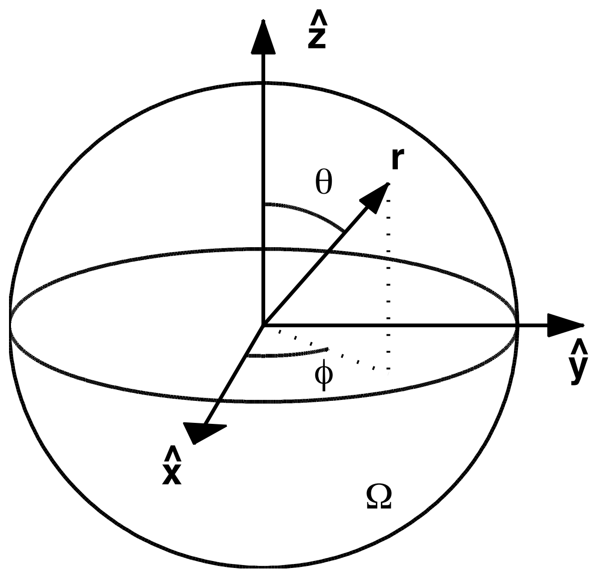


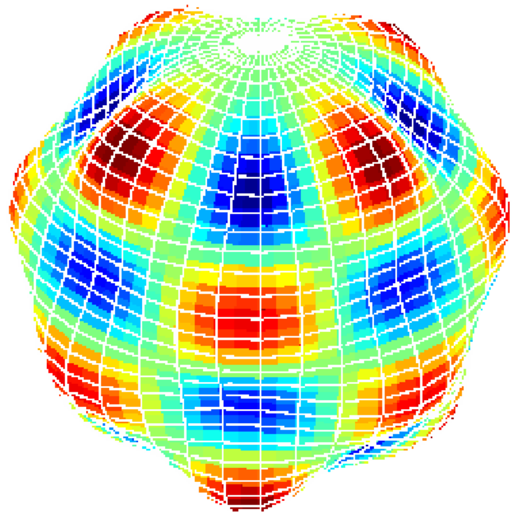
CHAMP & Earth's Magnetic Field — 2



Common problems — 1

Scalar **data** $d(\mathbf{r})$ modeled on a unit sphere Ω parameterized as $\mathbf{r} = (\theta, \phi)$:



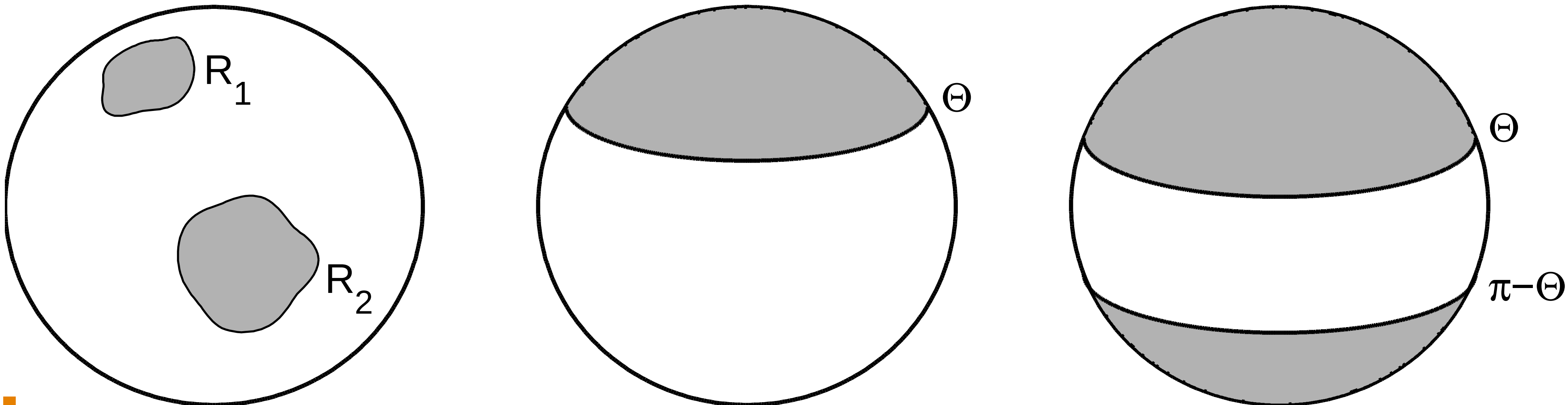


Spherical harmonics $Y_{lm}(\mathbf{r})$ form an **orthonormal** basis on Ω:

$$\int_{\Omega} Y^*_{lm} Y_{l'm'} \, d\Omega = \delta_{ll'} \delta_{mm'}.$$

← a delta function

The domain of **data availability** or the **region of interest** is $R \in \Omega$.



The spherical harmonics $Y_{lm}(\mathbf{r})$ are **not orthogonal** on R:

$$\boxed{\int_R Y_{lm}^* Y_{l'm'}\, d\Omega = D_{lm,l'm'}.}$$

← not a delta function

The *spatiospectral localization kernel* D is **not sparse**, but it is **blocky** (order m is a **good quantum number**) for axially **symmetric** R.

Slepian's problem

Eigenvectors of D are expansion coefficients of **Slepian functions**,

$$g(\mathbf{r}) = \sum_{lm}^{L} g_{lm} Y_{lm}(\mathbf{r}).$$

They satisfy the **spherical concentration problem** to the region R of area A:

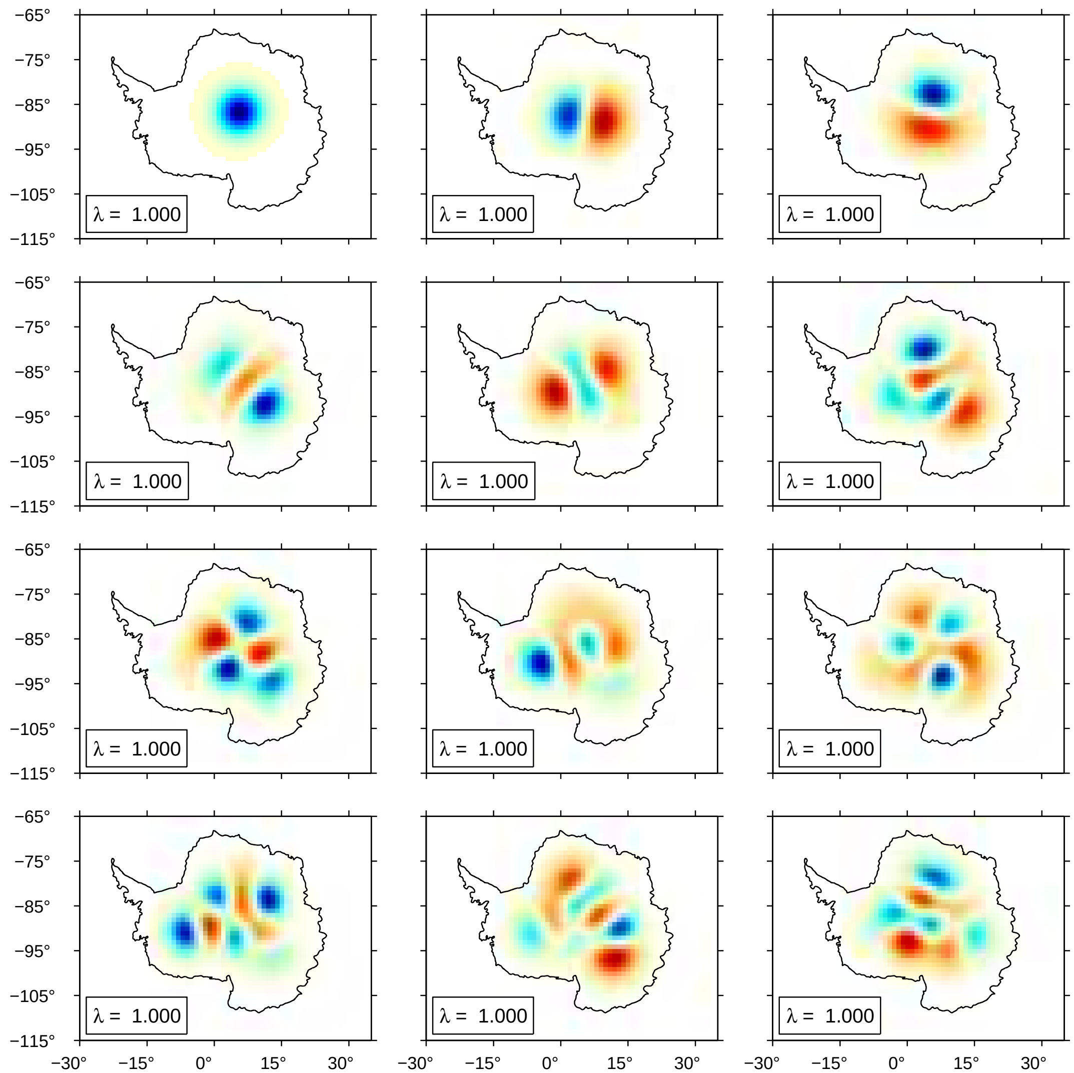
$$\lambda = \int_R g^2\, d\Omega \Bigg/ \int_\Omega g^2\, d\Omega = \text{maximum.}$$

The Slepian functions $g_\alpha(\mathbf{r})$, designed for *any* region R, are still **orthonormal** over the whole sphere Ω but now they are *also* **orthogonal** over the region R:

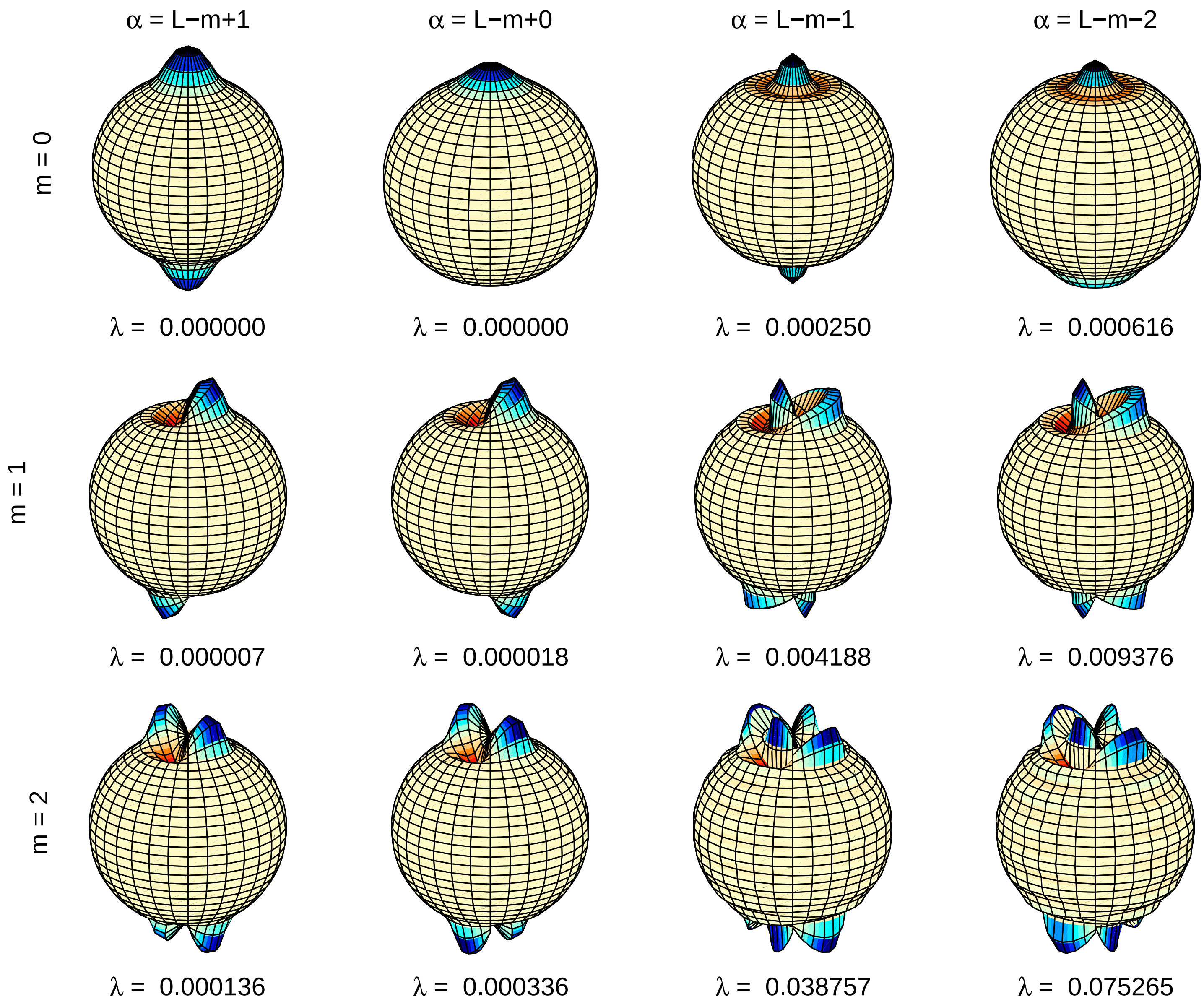
$$\boxed{\int_R g_\alpha g_\beta\, d\Omega = \lambda_\alpha \delta_{\alpha\beta} \quad \text{and} \quad \int_\Omega g_\alpha g_\beta\, d\Omega = \delta_{\alpha\beta}.}$$

They are a **doubly orthogonal bandlimited basis** for *localized* and *global* signals.

Slepian functions for abitrary regions $(L = 60)$



Slepian functions for cosmology



Signal estimation: **Problem 1**

Given $d(\mathbf{r})$ and $\langle n(\mathbf{r})n(\mathbf{r}')\rangle$, estimate the signal $s(\mathbf{r})$ at **source level**:

realizing that the estimate $\hat{s}(\mathbf{r})$ is **always bandlimited** to $0 \leq L < \infty$.

Spectral estimation: **Problem 2**

Given $d(\mathbf{r})$ and $\langle n(\mathbf{r})n(\mathbf{r}')\rangle$, and assuming the signal behaves as

estimate the **power spectral density** S_l, for $0 \leq l < \infty$.

Common problems — 3

Signal estimation: **Problem 1**

Given $d(\mathbf{r})$ and $\langle n(\mathbf{r})n(\mathbf{r}')\rangle$, estimate the signal $s(\mathbf{r})$ at **source level**:

$$\hat{s}(\mathbf{r}) = \sum_{lm}^{L} \hat{s}_{lm} Y_{lm}(\mathbf{r}),$$

realizing that the estimate $\hat{s}(\mathbf{r})$ is **always bandlimited** to $0 \leq L < \infty$.

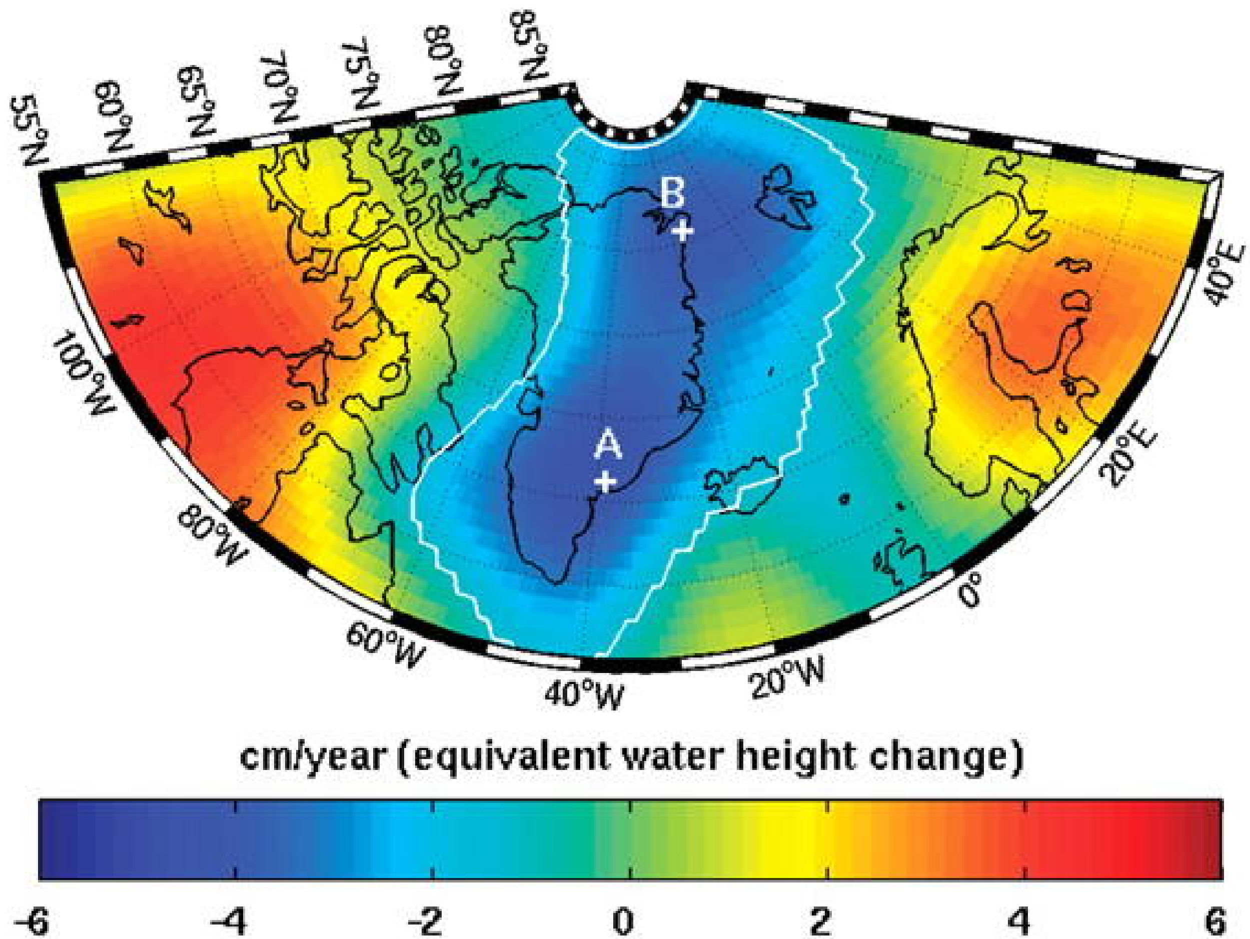
Spectral estimation: **Problem 2**

Given $d(\mathbf{r})$ and $\langle n(\mathbf{r})n(\mathbf{r}')\rangle$, and assuming the signal behaves as

$$\langle s_{lm}\rangle = 0 \quad \text{and} \quad \langle s_{lm} s^*_{l'm'}\rangle = S_l\, \delta_{ll'}\delta_{mm'},$$

estimate the **power spectral density** S_l, for $0 \leq l < \infty$.

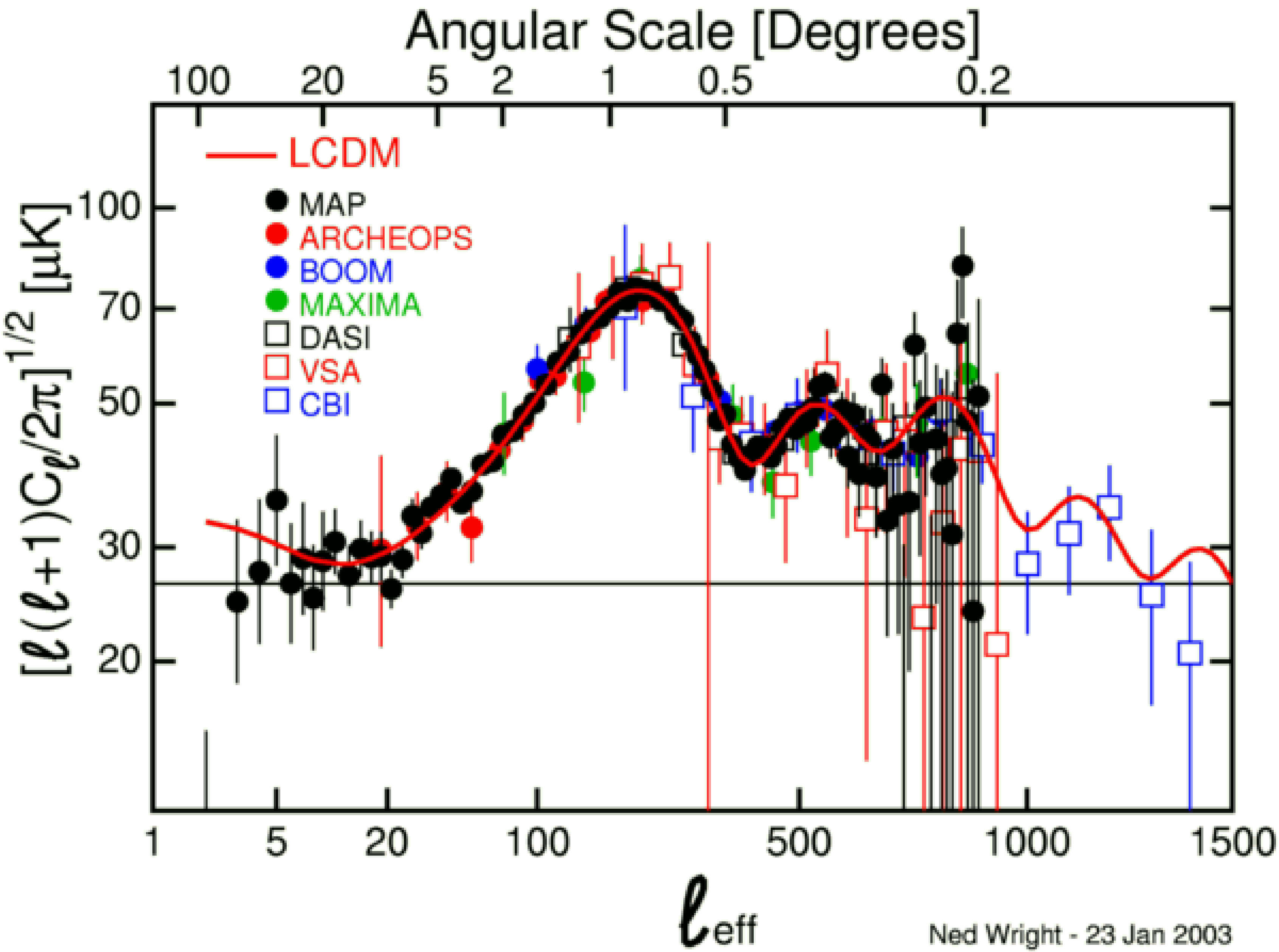
Why we solve Problem 1



Chen, Wilson & Tapley, *Science* (2006):

Spatial leakage effects are also evident, because of filtering applied to suppress the noise in high-degree and high- order spherical harmonics.

Why we solve Problem 2



Problem 1 — Finding the *signal*

Construct a **bandlimited estimate** in the spherical harmonic basis by minimizing the **quadratic misfit to the data** over R. The optimal solution depends on $\mathbf{D}^{-1}$:

$$\hat{s}_{lm} = \sum_{l'm'}^{L} D^{-1}_{lm,l'm'} \int_R d\, Y^*_{l'm'}\, d\Omega.$$

Finding $\mathbf{D}^{-1}$ is tough, so construct a **truncated-Slepian basis** estimate instead:

The solution depends on the localization **eigenvalue** λ_α **at the same rank**:

Truncation prevents the blowup of the low eigenvalues.

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The solution depends on the localization **eigenvalue** λ_α **at the same rank**:

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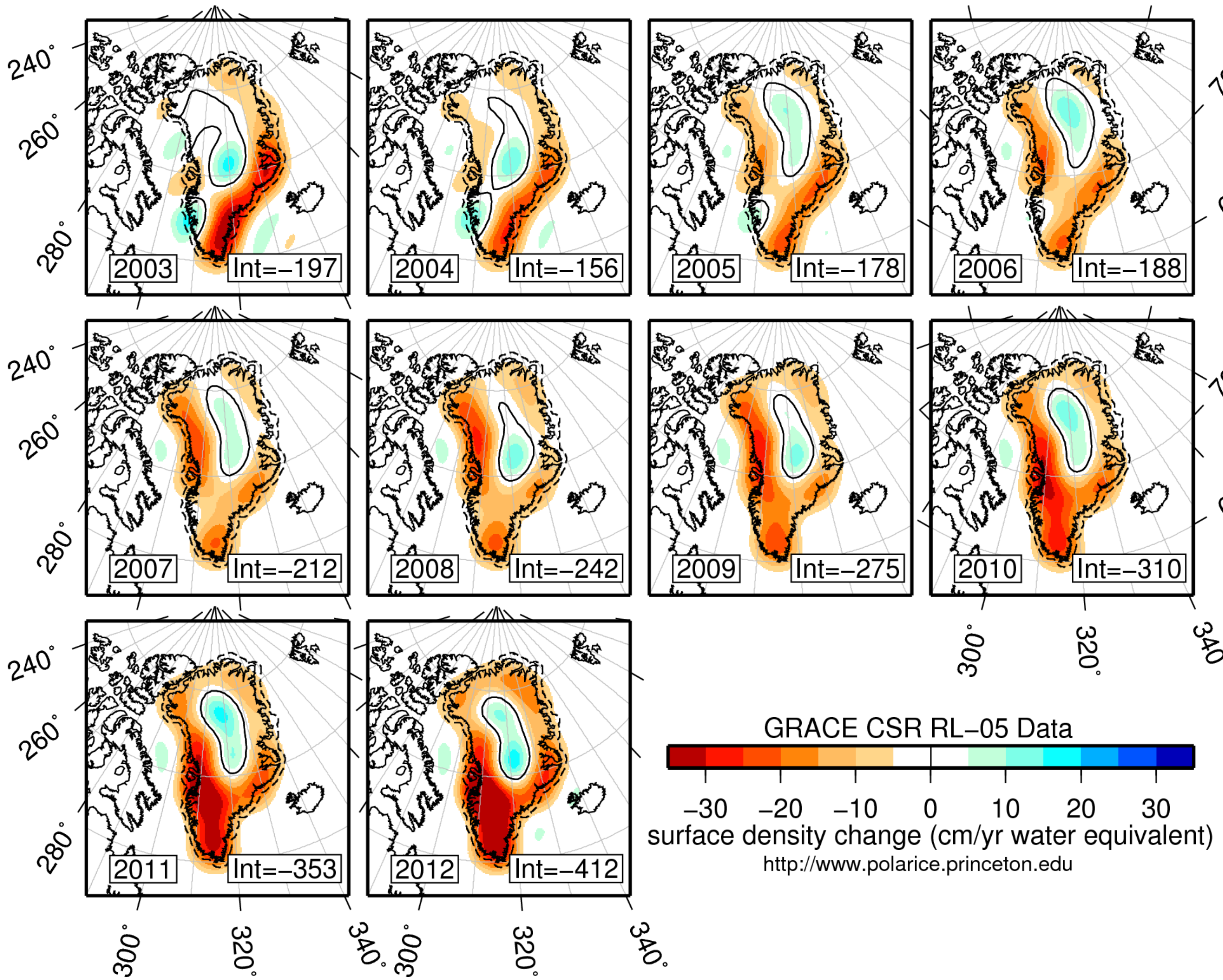
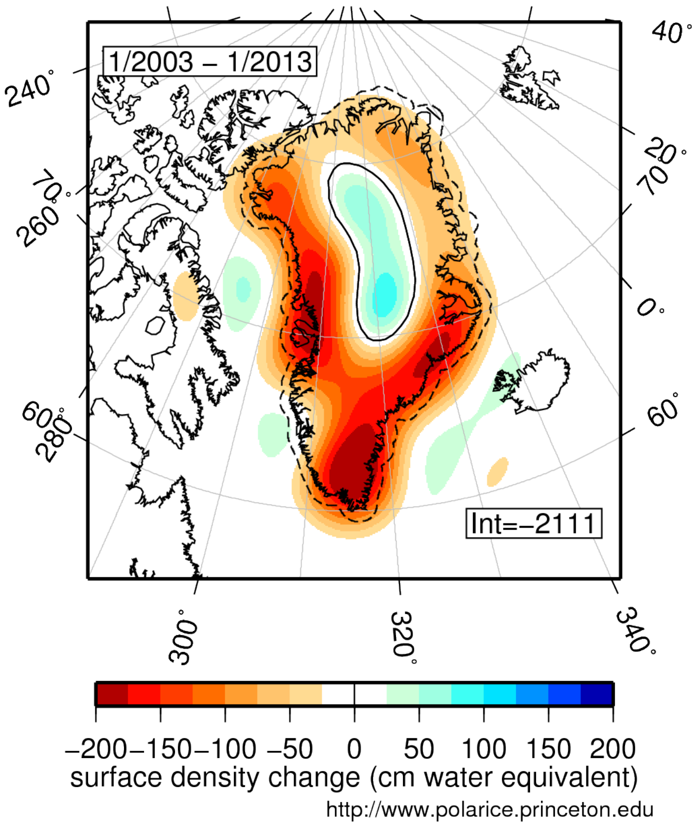
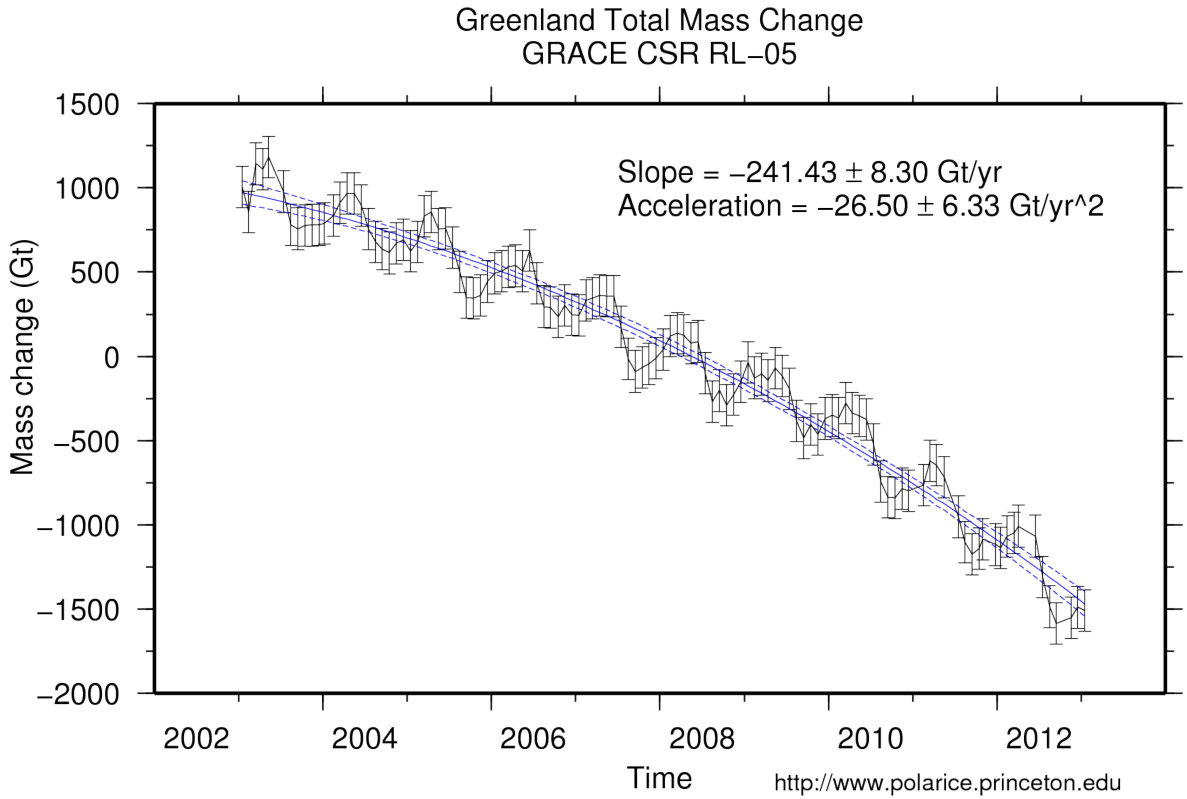
Truncation prevents the blowup of the low eigenvalues.

Slepian estimation of gravity field changes — 1

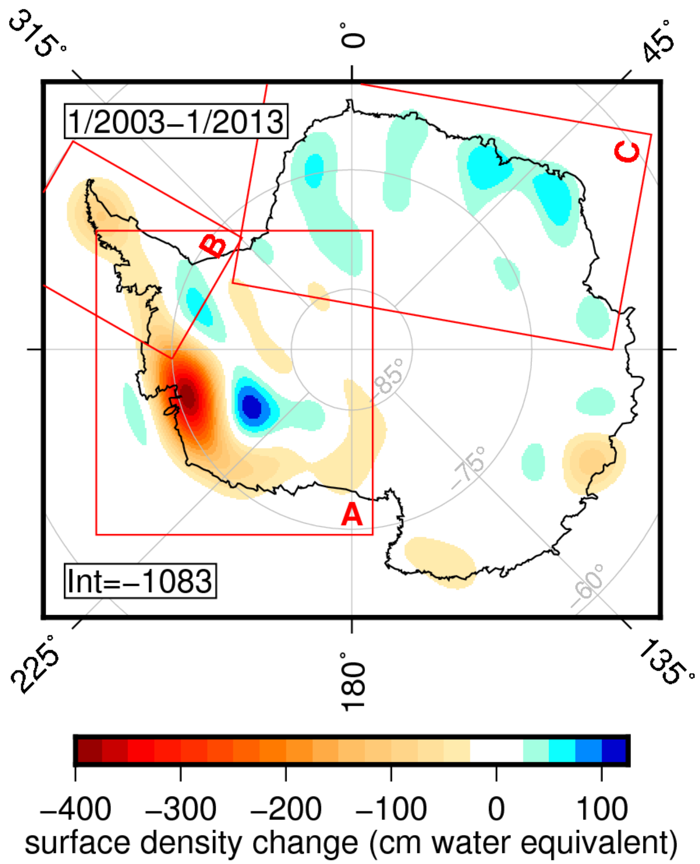


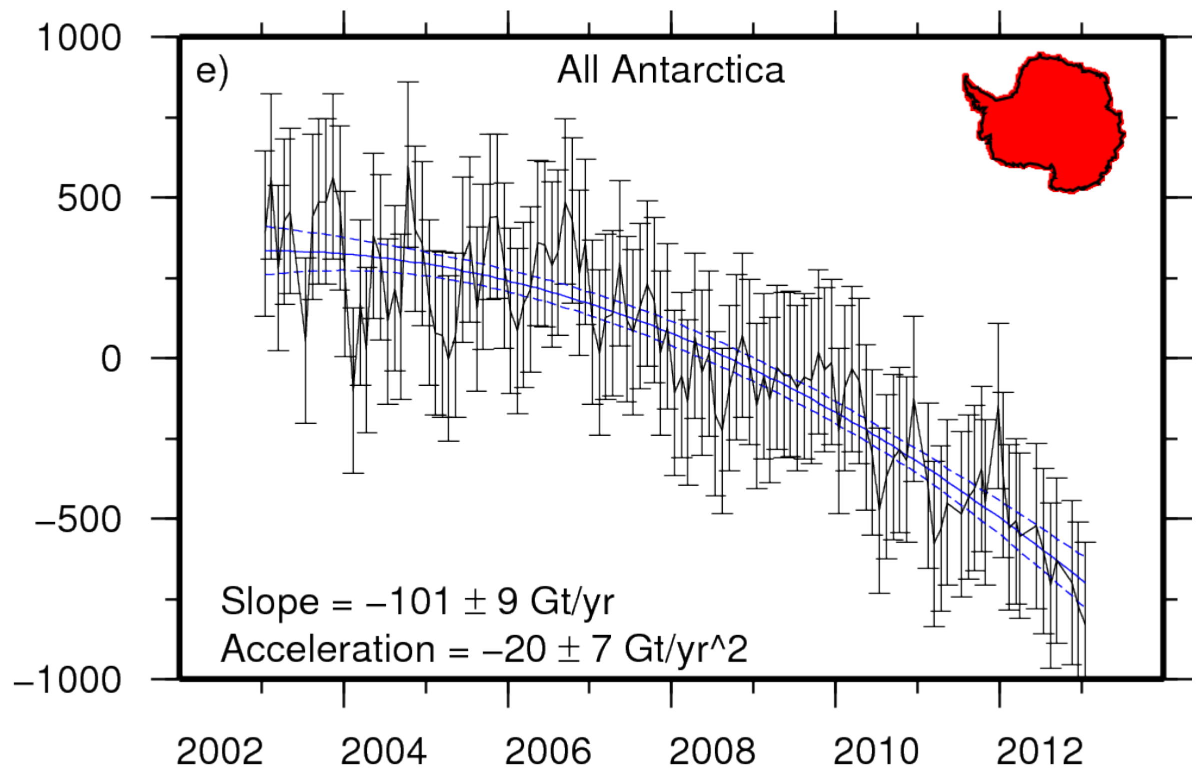
Slepian estimation of gravity field changes — 2



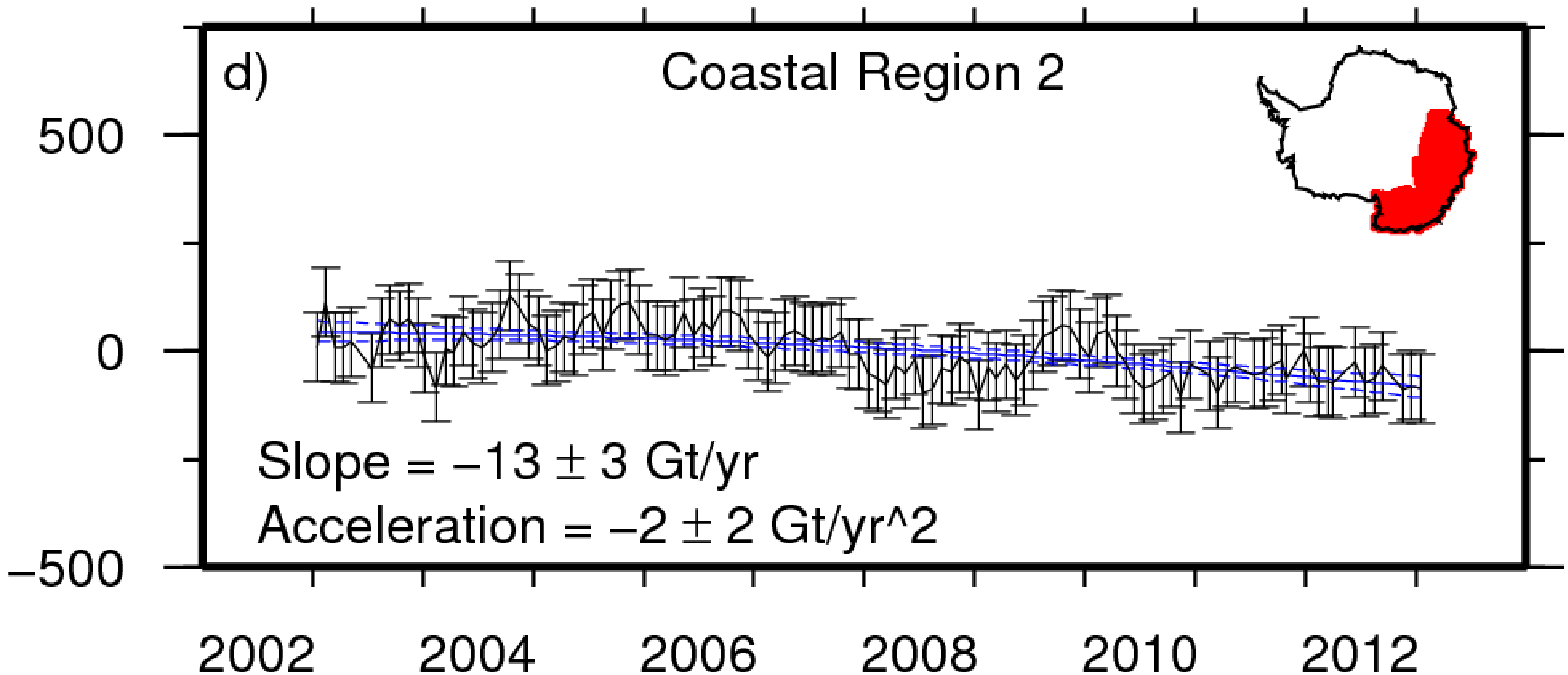


Slepian estimation of gravity field changes — 3

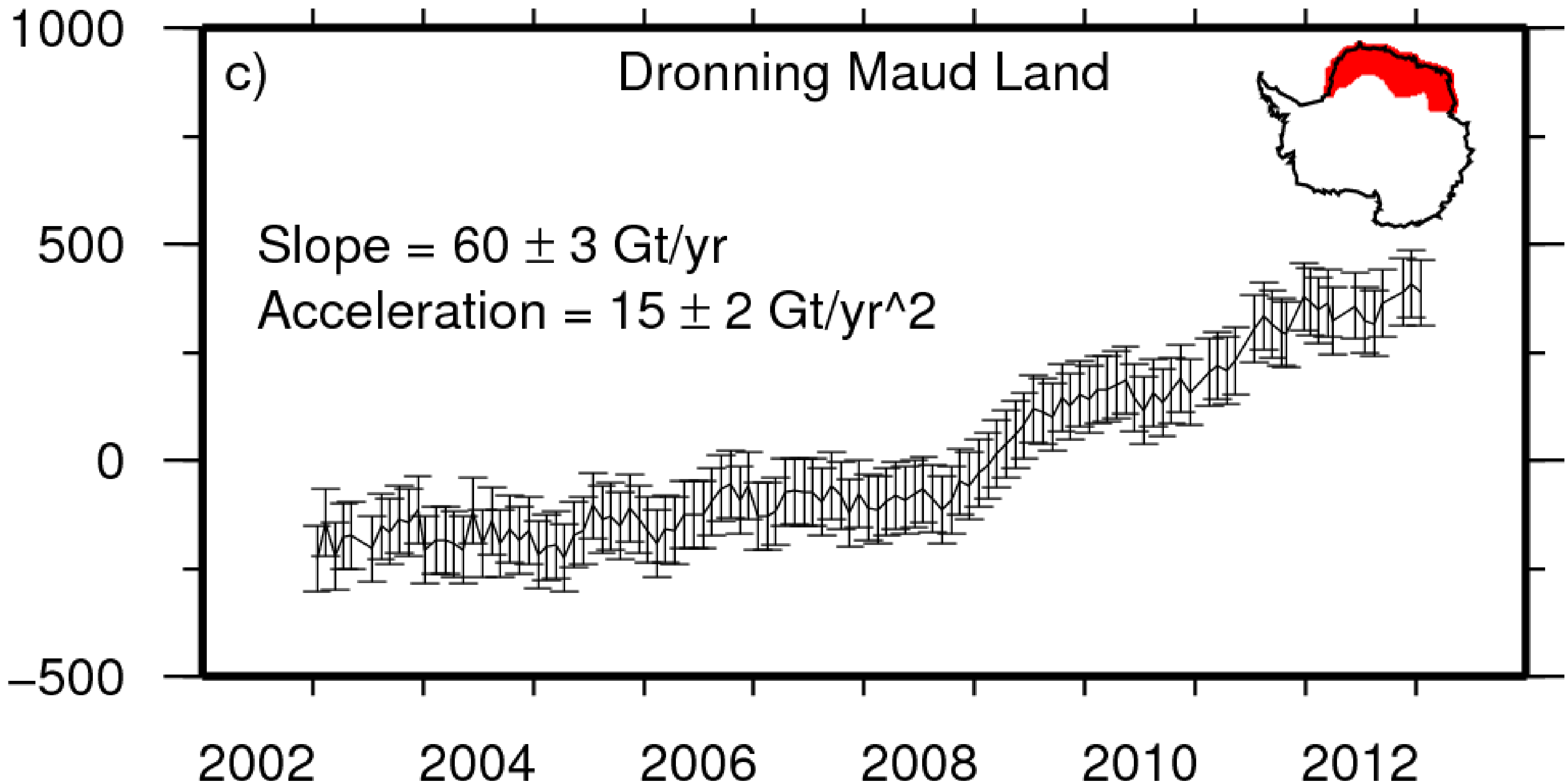




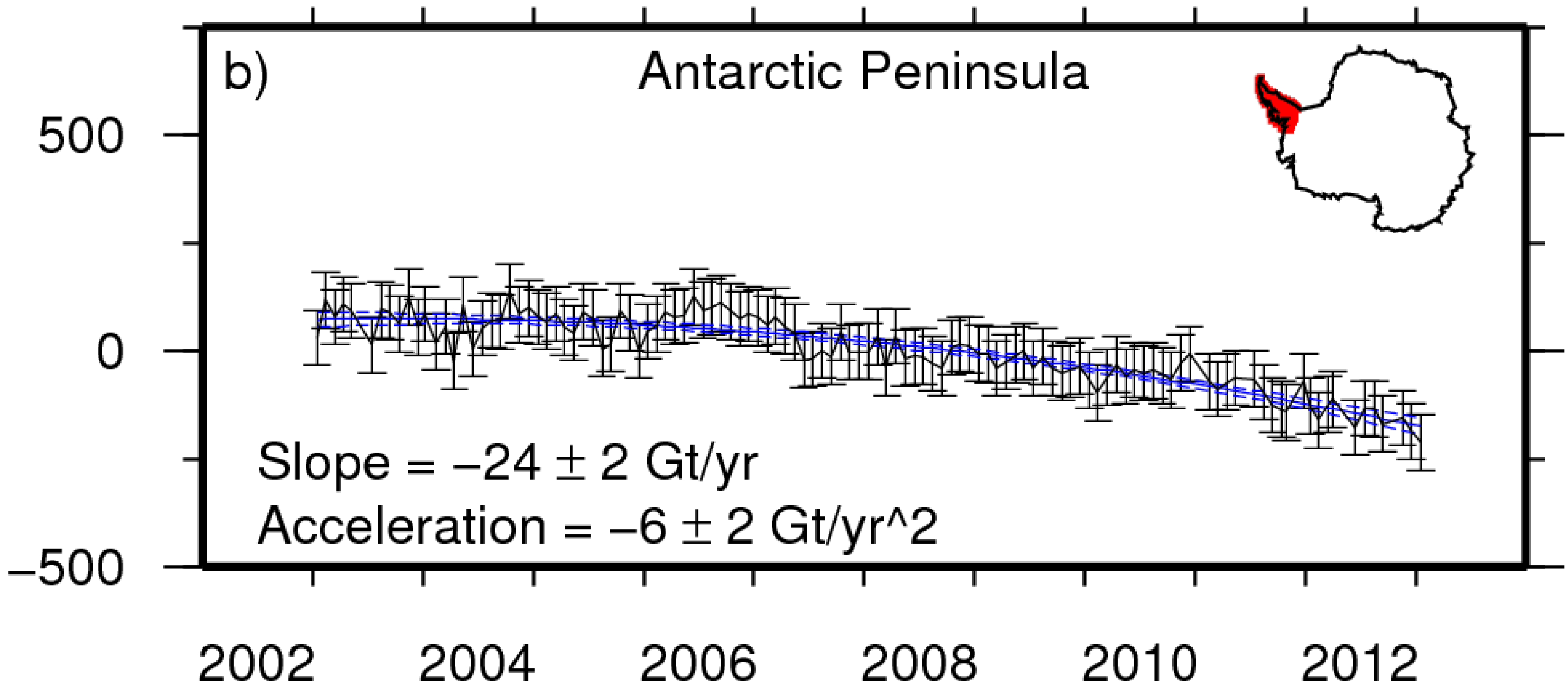
Ice mass loss (2002–2013) — 1



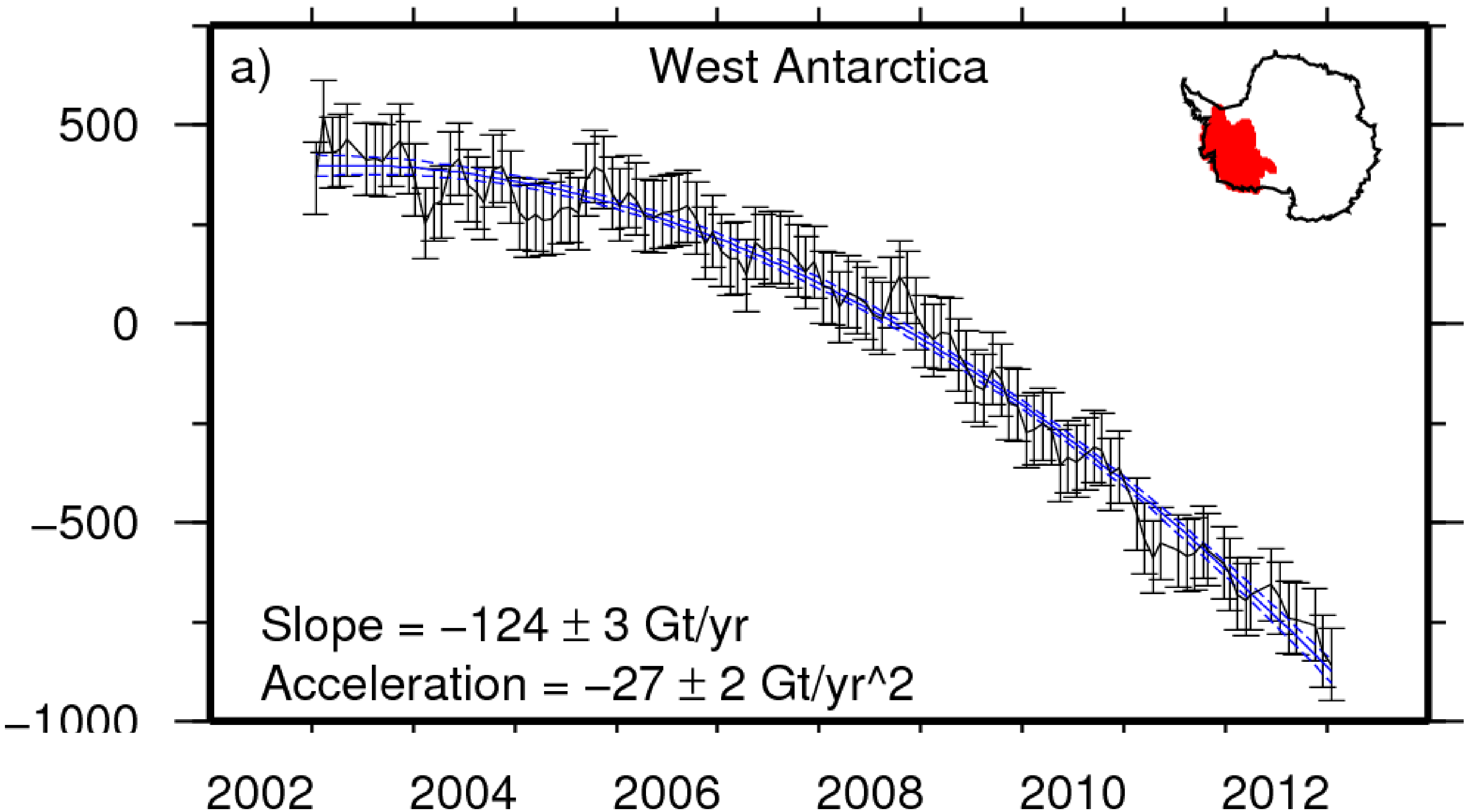
Ice mass loss (2002–2013) — 2



Ice mass loss (2002–2013) — 3



Ice mass loss (2002–2013) — 4



High-quality spectral estimation

To assess the quality of our spectral estimates $\hat{S}_l$, we calculate:

$$\begin{aligned} \text{variance:} \quad v_l &= \langle \hat{S}_l^2 \rangle - \langle \hat{S}_l \rangle^2 && (1) \\ \text{bias:} \quad b_l &= \langle \hat{S}_l \rangle - S_l && (2) \\ \text{error:} \quad \epsilon_l &= \hat{S}_l - S_l && (3) \\ \text{mse:} \quad \langle \epsilon_l^2 \rangle &= v_l + b_l^2. && (4) \end{aligned}$$

A good estimator is unbiased and/or minimizes the mse.

The industry-standard **maximum-likelihood method** via the iterative, nonlinear, **Newton-Raphson** algorithm returns the **minimum-variance unbiased estimate** of the power spectral density — but the estimation variance is quite high!:

$$\langle \hat{S}_l^{\mathrm{ML}} \rangle = S_l. \qquad (5)$$

Finding the *spectrum*, the *multitaper* way

Use the Slepian functions as data tapers, with weights be chosen iteratively to minimize the mse of the multitaper estimate (*Wieczorek & Simons*, JFAA, 2007). *Dahlen & Simons, GJI (2008)* choose the **eigenvalues** of $\mathbf{D}$:

$$\hat{S}_l^{\mathrm{MT}} = \frac{1}{K}\sum_{\alpha} \lambda_\alpha \left(\frac{1}{2l+1} \sum_m \left| \int_\Omega g_\alpha(\mathbf{r})\, d(\mathbf{r})\, Y_{lm}^*(\mathbf{r})\, d\Omega \right|^2 \right)$$

Bias (degree coupling) depends only on the bandwidth L of the Slepian windows, and **variance** is almost exactly K **times smaller than the periodogram variance** when their (effective) bandwidths are similar, as it is in the 1-D case!

Spectral and spatial concentration trade off via the **Shannon number**, which is the sole parameter to be chosen by the analyst:

$$K = \sum_{\alpha}^{(L+1)^2} \lambda_\alpha = (L+1)^2 \frac{A}{4\pi}.$$

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Sorry, wrong journal

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GJI Geodesy, potential field and applied geophysics

Spectral estimation on a sphere in geophysics and cosmology

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SUMMARY

We address the problem of estimating the spherical-harmonic power spectrum of a statistically isotropic scalar signal from noise-contaminated data on a region of the unit sphere. Three different methods of spectral estimation are considered: (i) the spherical analogue of the one-dimensional (1-D) periodogram, (ii) the maximum-likelihood method and (iii) a spherical analogue of the 1-D multitaper method. The periodogram exhibits strong spectral leakage, especially for small regions of area $A \ll 4\pi$, and is generally unsuitable for spherical spectral analysis applications, just as it is in 1-D. The maximum-likelihood method is particularly useful in the case of nearly-whole-sphere coverage, $A \approx 4\pi$, and has been widely used in cosmology to estimate the spectrum of the cosmic microwave background radiation from spacecraft observations. The spherical multitaper method affords easy control over the fundamental trade-off between spectral resolution and variance, and is easily implemented regardless of the region size, requiring neither non-linear iteration nor large-scale matrix inversion. As a result, the method is ideally suited for most applications in geophysics, geodesy or planetary science, where the objective is to obtain a spatially localized estimate of the spectrum of a signal from noisy data within a pre-selected and typically small region.

Key words: Time series analysis; Fourier analysis; Inverse theory; Spatial analysis.

Whole-sphere spectral estimate

Assuming **isotropy**, add the power from all orders and subtract noise term:

$$\hat{S}_l^{\mathrm{WS}} = \frac{1}{2l+1} \sum_m \left| \int_\Omega d(\mathbf{r})\, Y_{lm}^*(\mathbf{r})\, d\Omega \right|^2 - N_l,$$

This estimate is **unbiased**:

$$b_l^{\mathrm{WS}} = 0, \tag{6}$$

and its **variance**, our **gold standard**, can be calculated from elementary statistics:

$$v_l^{\mathrm{WS}} = \frac{2}{2l+1} \left(S_l + N_l\right)^2, \tag{7}$$

In the absence of noise, the nonzero sampling variance is termed **cosmic**.

The problem is that **we do not have whole-sphere data**.

Cut-sphere spectral estimate (periodogram)

Simply work with the available data — i.e. use a gain-adjusted **boxcar window**:

$$\hat{S}_l^{\mathrm{SP}} = \left(\frac{4\pi}{A}\right)\frac{1}{2l+1}\sum_m \left|\int_R d(\mathbf{r})\,Y_{lm}^*(\mathbf{r})\,d\Omega\right|^2 - \text{noise correction.}$$

This estimate is **biased** (unless $S_l = S$ or $R = \Omega$):

$$b_l^{\mathrm{SP}} = \sum_{l'}\left[\left(\frac{4\pi}{A}\right)\frac{1}{2l+1}\sum_{mm'}\left|D_{lm,l'm'}\right|^2 - \delta_{ll'}\right] S_{l'}, \tag{8}$$

and the **variance** is:

$$v_l^{\mathrm{SP}} = \frac{2(4\pi/A)^2}{(2l+1)^2}\sum_{mm'}\left|\sum_{pq}(S_p+N_p)\left|D_{lm,pq}\right|^2\right|^2. \tag{9}$$

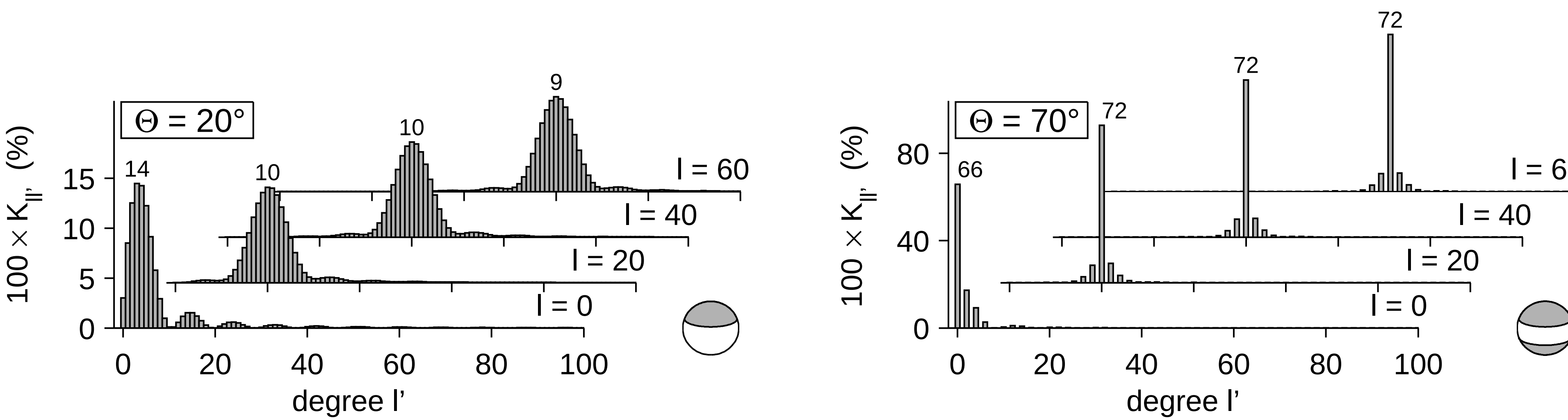
The appearance of the *spatiospectral localization kernel* D in these expressions has been known since at least the work of *Peebles and Hauser* (1973).

The periodogram coupling matrix

The periodogram coupling matrix shows the **leakage** from untargeted degrees:

$$K_{ll'} = \left(\frac{4\pi}{A}\right) \frac{1}{2l+1} \sum_{mm'} \left|D_{lm,l'm'}\right|^2 .$$

It shows us the contributions of adjacent l' when we seek the power at l.



This coupling is **neither bandlimited nor very well localized** in the spectrum...

we can do much better!

Single-taper spectral estimate

Use one of the Slepian functions as data window:

$$\hat{S}_l^{\alpha} = \frac{1}{2l+1} \sum_m \left| \int_\Omega g_\alpha(\mathbf{r})\, d(\mathbf{r})\, Y_{lm}^*(\mathbf{r})\, d\Omega \right|^2 - \text{noise correction}, \tag{10}$$

to obtain a **biased estimate** controlled by a **coupling matrix**

$$\boxed{M_{ll'}^{\alpha} = \left(\frac{2l'+1}{4\pi}\right) \sum_{pq} |g_{\alpha,pq}|^2 \begin{pmatrix} l & p & l \\ 0 & 0 & 0 \end{pmatrix}^2}$$

which shows the resulting estimate is a **bandlimited and well-localized** average of the spectral power within a certain bandwidth.

Spectral and spatial concentration trade off via the **Shannon number**, which is the sole parameter to be chosen by the analyst.

As with the periodogram, the Slepian functions can be normalized to yield an **unbiased** estimate of a **white** power spectrum.

Multiple-taper spectral estimate

Single-tapers are good, but **weighted averaging** the estimates made with many different tapers **reduces the estimation variance** without increasing the bias.

$$\hat{S}_l^{\mathrm{MT}} = \sum_{\alpha} c_\alpha \hat{S}_l^{\alpha} \qquad \text{where} \qquad \sum_{\alpha} c_\alpha = 1. \tag{11}$$

Two tapers α and β have a **covariance** that behaves *almost* as if both estimates were **statistically uncorrelated**:

$$v_l^{\alpha\beta} = \text{diagonally dominant}, \tag{12}$$

thus the **multitaper estimation variance**

$$v_l^{\mathrm{MT}} = \sum_{\alpha\beta} c_\alpha \, v_l^{\alpha\beta} c_\beta \tag{13}$$

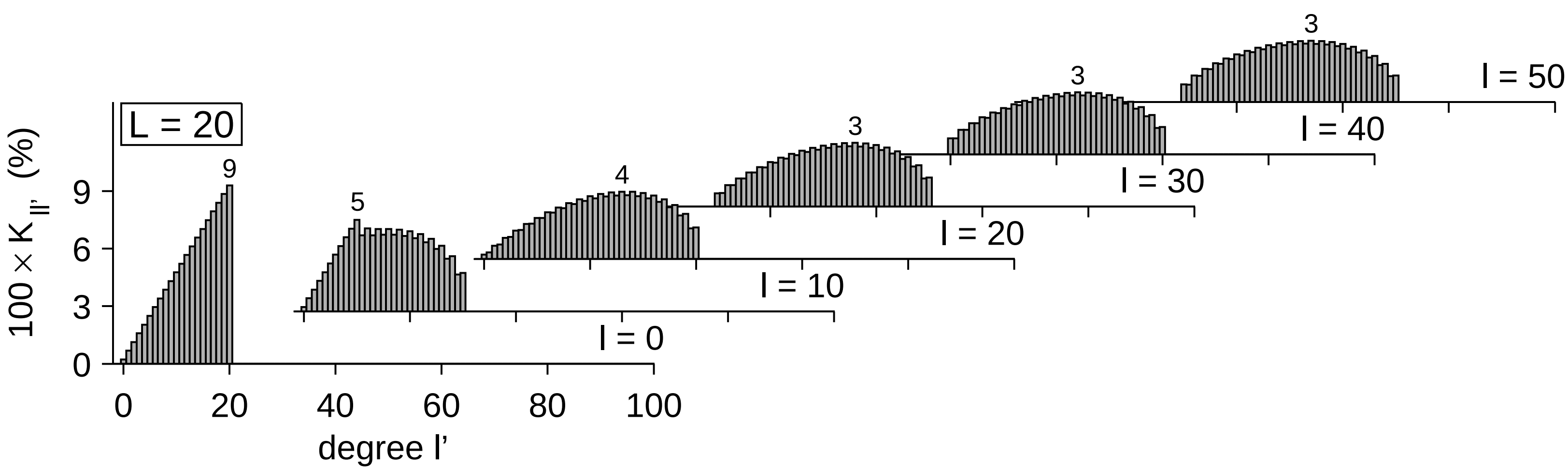
is **reduced** by the addition of estimates made with subsequent tapers.

The multitaper coupling matrix

The weights c_α can be chosen iteratively to minimize the mean-squared error of the multitaper estimate. However, a logical choice are the **eigenvalues** of D. Then the **multitaper coupling matrix** is

$$M_{ll'} = \frac{2l'+1}{(L+1)^2} \sum_p^L (2p+1) \begin{pmatrix} l & p & l' \\ 0 & 0 & 0 \end{pmatrix}^2,$$

which — amazingly — depends only upon the chosen bandwidth L.



Quadratic spectral estimators

Maximum-likelihood ... *very cumbersome, unbiased, high variance*

Whole-sphere ... *unattainable*

$$\hat{S}_l^{\mathrm{WS}} = \frac{1}{2l+1} \sum_m \left| \int_\Omega d(\mathbf{r})\, Y_{lm}^*(\mathbf{r})\, d\Omega \right|^2 - \text{noise correction.} \tag{14}$$

Periodogram ... *broadband bias, high variance*

$$\hat{S}_l^{\mathrm{SP}} = \left(\frac{4\pi}{A}\right) \frac{1}{2l+1} \sum_m \left| \int_R d(\mathbf{r})\, Y_{lm}^*(\mathbf{r})\, d\Omega \right|^2 - \text{noise correction.} \tag{15}$$

Single-taper ... *bandlimited bias*

$$\hat{S}_l^{\alpha} = \frac{1}{2l+1} \sum_m \left| \int_\Omega g_\alpha(\mathbf{r})\, d(\mathbf{r})\, Y_{lm}^*(\mathbf{r})\, d\Omega \right|^2 - \text{noise correction.} \tag{16}$$

Multiple-taper ... *bandlimited bias, lower variance, easily implemented*

$$\hat{S}_l^{\mathrm{MT}} = \frac{1}{K} \sum_\alpha \lambda_\alpha \hat{S}_l^{\alpha}. \tag{17}$$

Appraisal of performance

We study the performance of the various estimators by forming the variance ratios

$$\boxed{(\sigma_l^2)^{\mathrm{XX}} = v_l^{\mathrm{XX}}/v_l^{\mathrm{WS}}}$$

where XX stands for any of the acronyms SP, DP, ML or MT.

The **deconvolved periodogram**

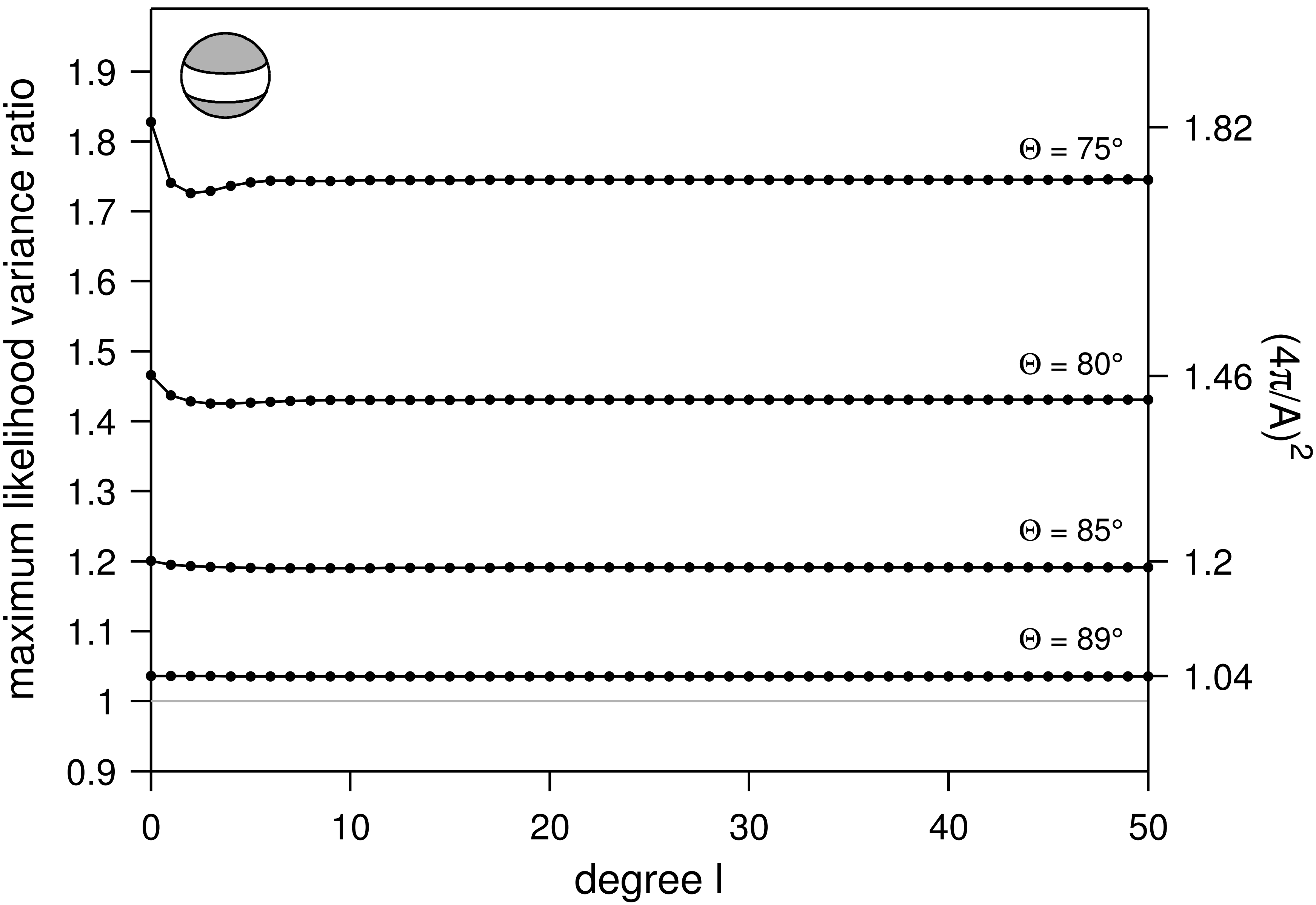
$$\hat{S}_l^{\mathrm{DP}} = \sum_{l'} K_{ll'}^{-1} \hat{S}_{l'}^{\mathrm{SP}}, \tag{18}$$

where K is the **periodogram coupling matrix**, is **unbiased** $\langle \hat{S}_l^{\mathrm{DP}} \rangle = S_l$.

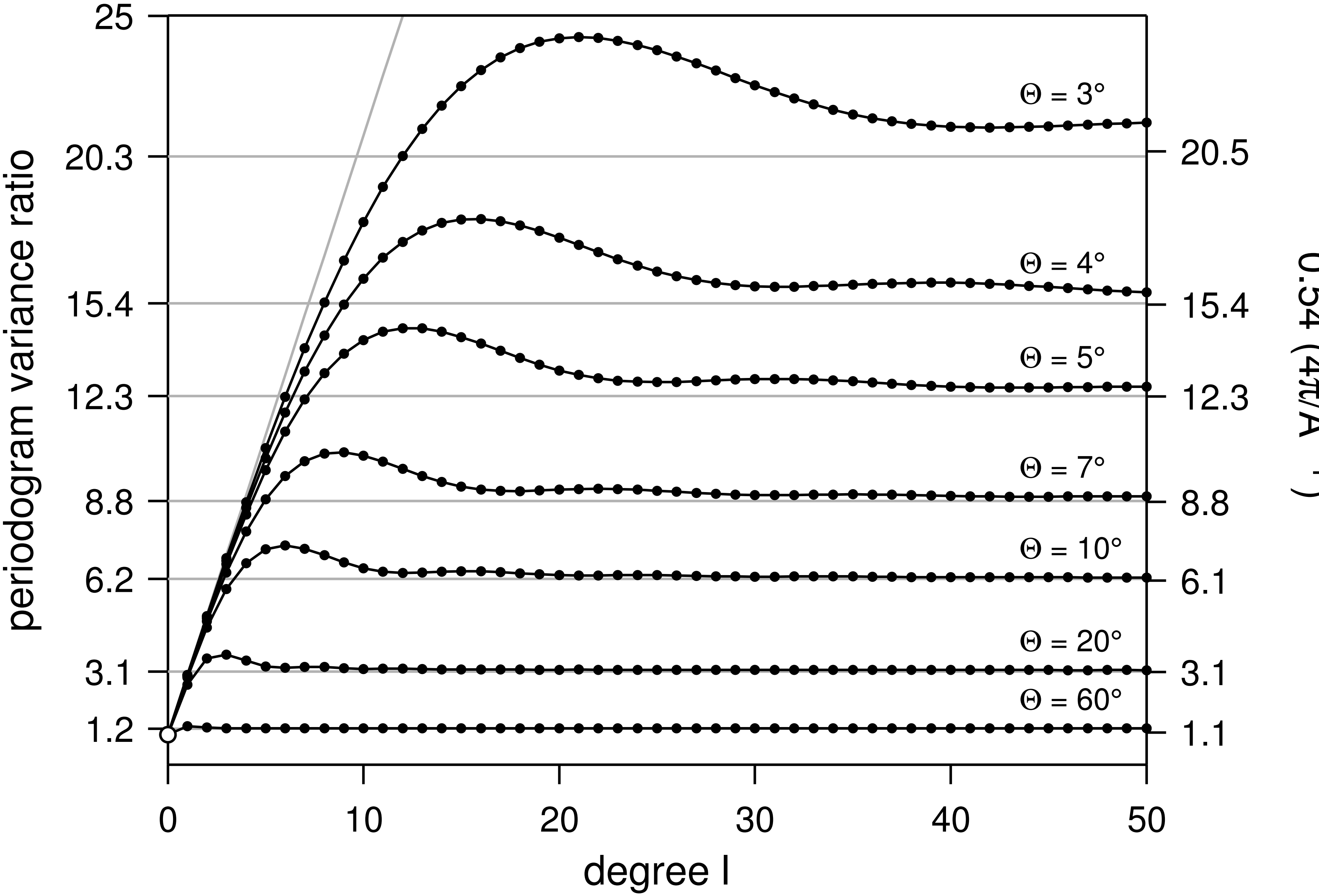
For **white** signal and noise, $S_l = S$ and $N_l = N$, the deconvolved periodogram concides with the maximum likelihood estimator.

Since $(\sigma_l^2)^{\mathrm{ML}} = (\sigma_l^2)^{\mathrm{DP}} = \left(\frac{4\pi}{A}\right) K_{ll}^{-1}$, we can compute the maximum-likelihood variance ratio without actually forming the maximum-likelihood estimate.

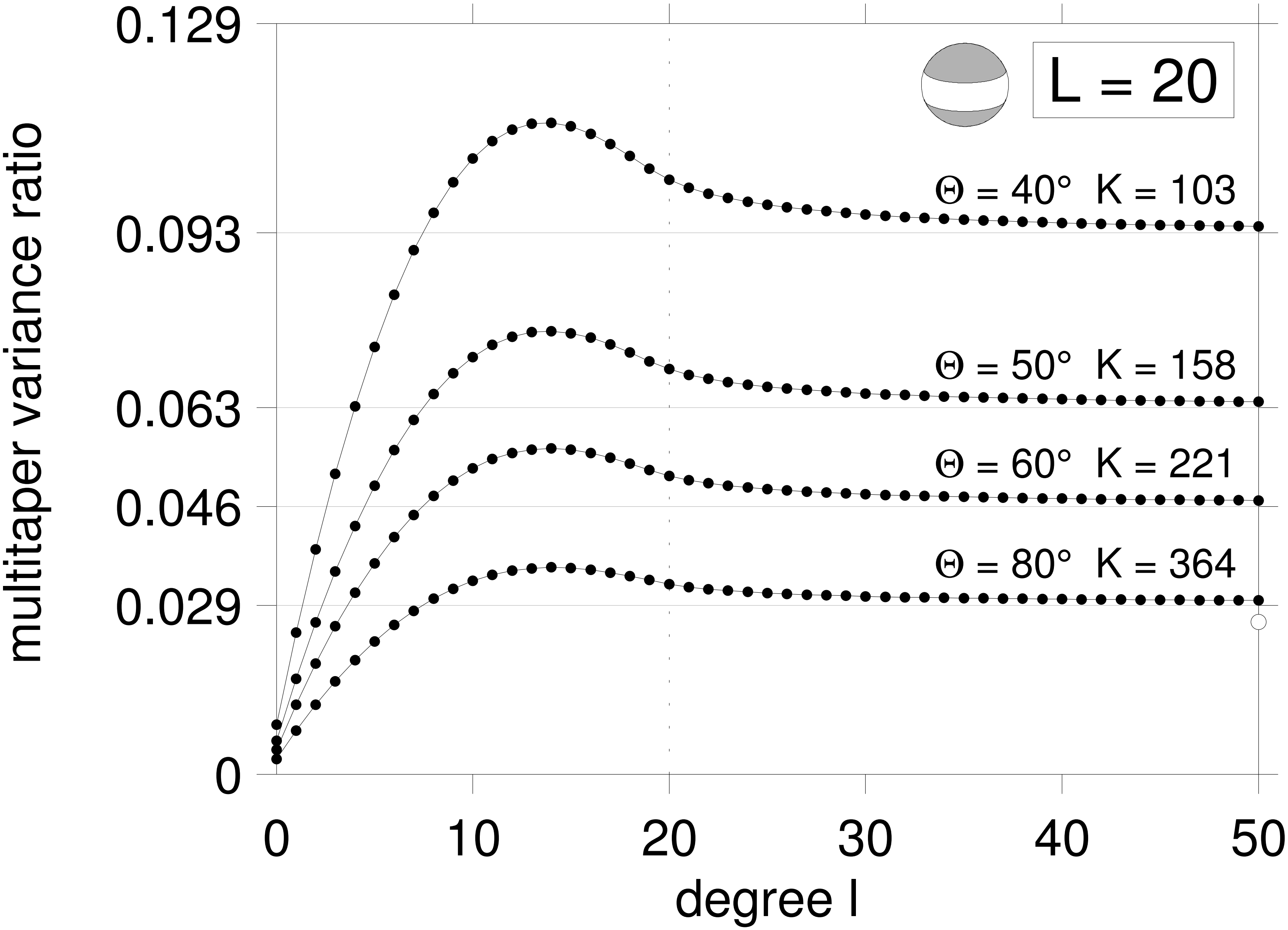
Maximum-likelihood / Whole-sphere variance



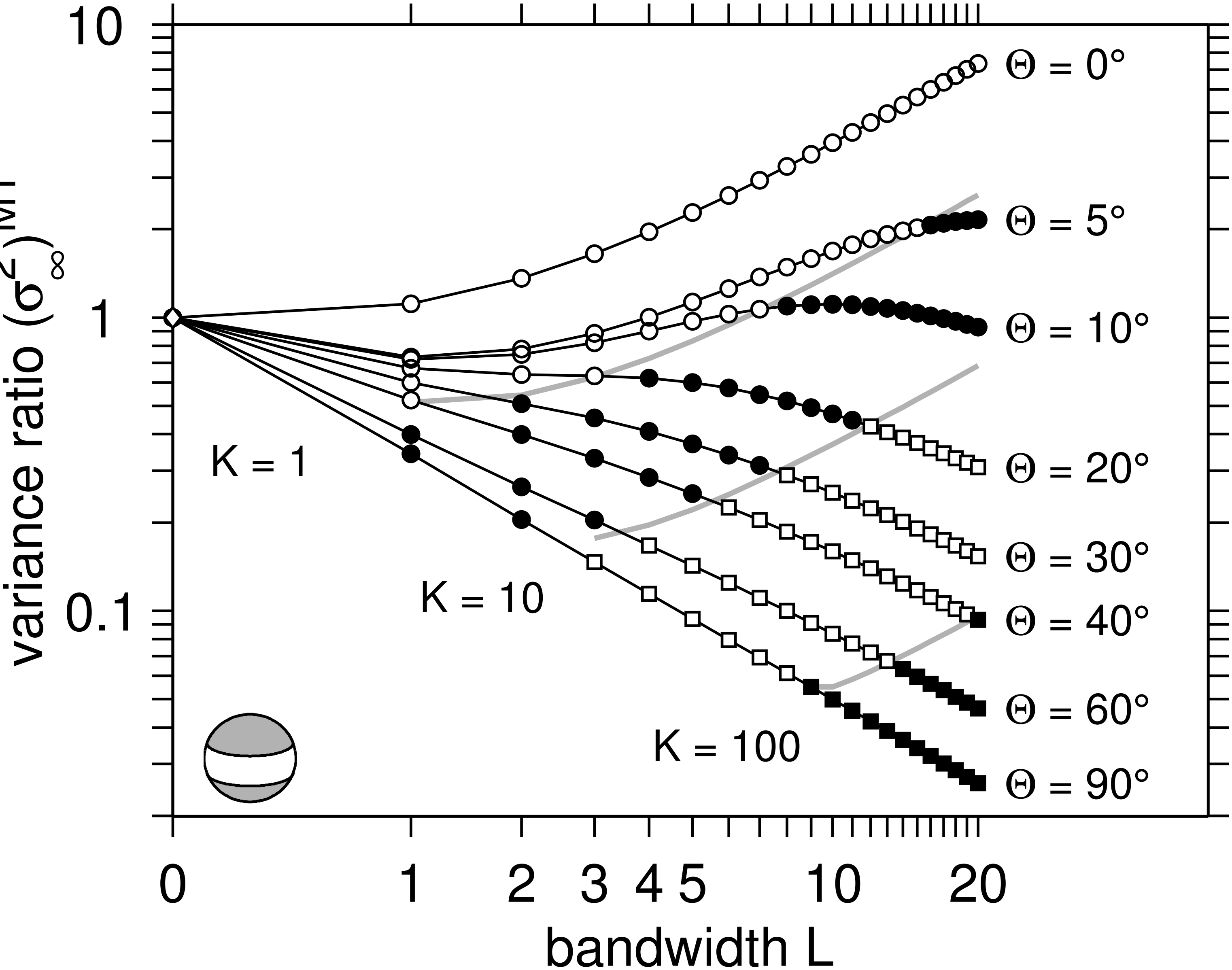
Periodogram / Whole-sphere variance



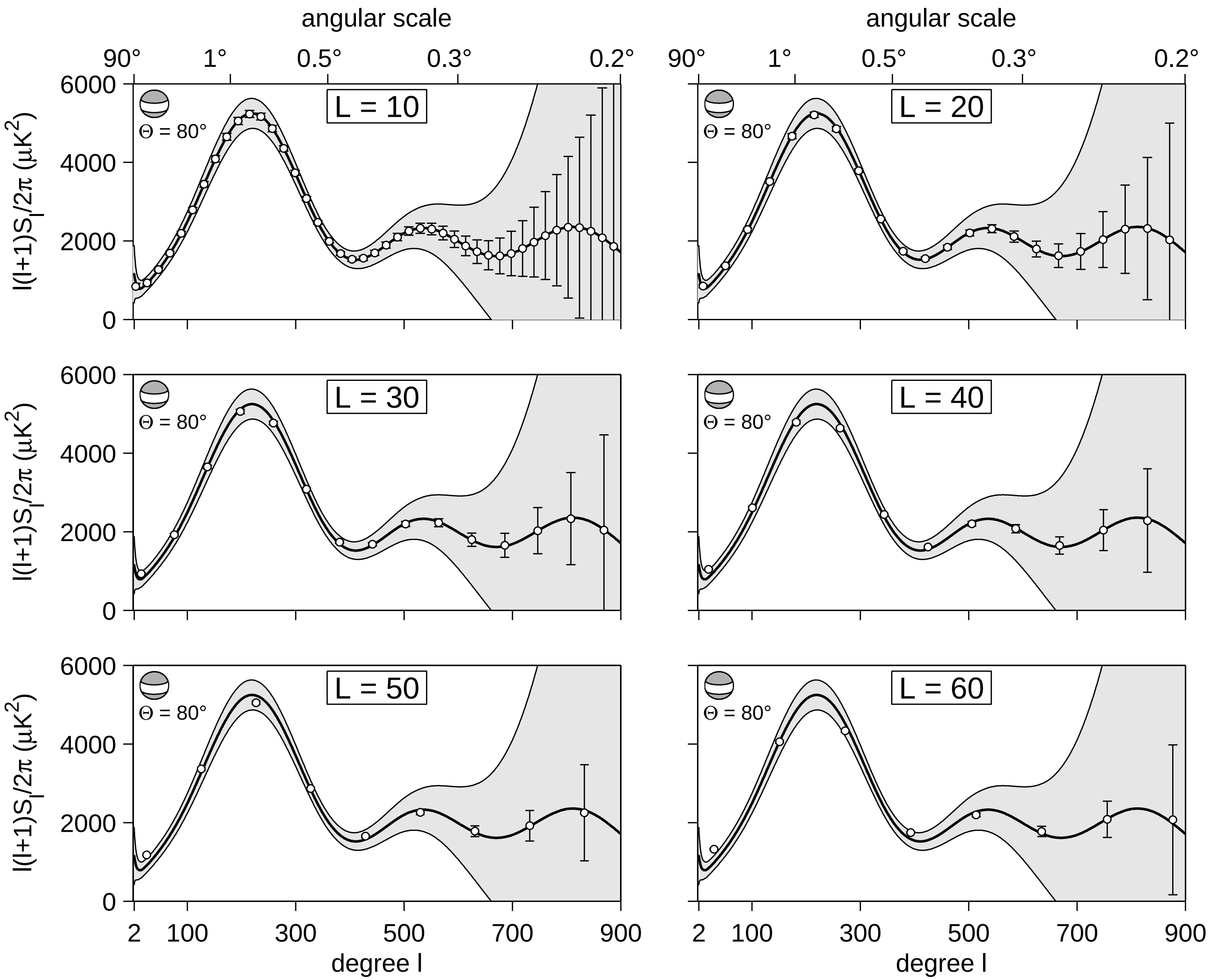
Multitaper / Whole-sphere variance



Multitaper / Whole-sphere variance at large l



An example for cosmology



Conclusions

1. Each the discussed estimators is **quadratic** in the data: a common framework allows us to study their relative merits.

2. The **maximum-likelihood estimate** provides the best unbiased estimate of the spectrum. However, its calculation is cumbersome, and requires an iterative procedure and the inversion of very large matrices.

3. The **periodogram estimate** is generally unsuitable for power spectral estimation on the sphere, much like it is in one dimension.

4. The **Slepian multitaper method** yields a smoothed and thus **biased** estimate of the spectrum, but it requires neither iteration nor large-scale matrix inversion. Its **variance is much lower** than that of any other method, and the only parameter that needs to be specified by the analyst is the **Shannon number**, or the space-bandwidth product diagnostic of the spatiospectral concentration.

Review: Scalar Slepian functions

Eigenvectors of $\mathbf{D}$ expand to **bandlimited Slepian functions**:

that satisfy **Slepian's concentration problem** to the region R of area A:

The **Shannon number**, or sum of the eigenvalues,

is the **effective dimension** of the space for which the bandlimited g are a **basis.**

Review: Scalar Slepian functions

Eigenvectors of $\mathbf{D}$ expand to **bandlimited Slepian functions**:

$$g = \sum_{lm}^{L} g_{lm} Y_{lm},$$

that satisfy **Slepian's concentration problem** to the region R of area A:

$$\lambda = \int_R g^2 \, d\Omega \Bigg/ \int_\Omega g^2 \, d\Omega = \text{maximum.}$$

The **Shannon number**, or sum of the eigenvalues,

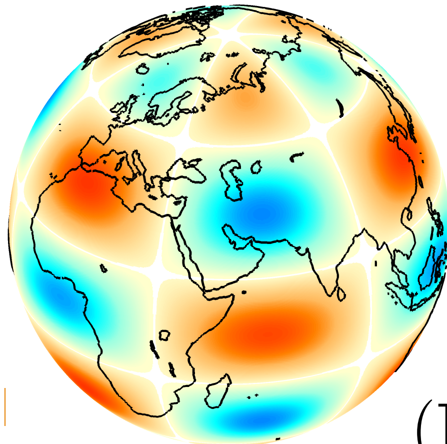
$$K = (L+1)^2 \frac{A}{4\pi},$$

is the **effective dimension** of the space for which the bandlimited g are a **basis**.

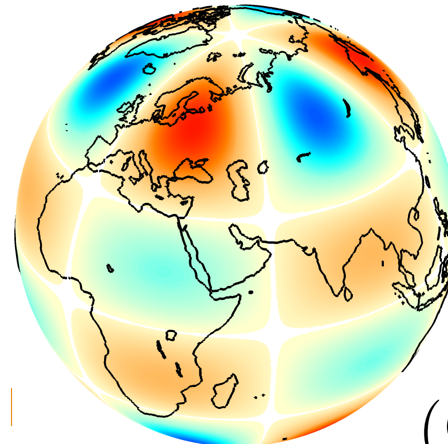
Voilà! We have *concentrated* a poorly localized basis of $(L+1)^2$ functions, Y_{lm}, both *spatially* and *spectrally*, to a new basis with only about K functions, g.

Vector spherical harmonics

$$\mathbf{B}_{lm} = \frac{\boldsymbol{\nabla}_1 Y_{lm}}{\sqrt{l(l+1)}} \qquad \mathbf{C}_{lm} = \frac{-\hat{\mathbf{r}}\times\boldsymbol{\nabla}_1 Y_{lm}}{\sqrt{l(l+1)}}$$

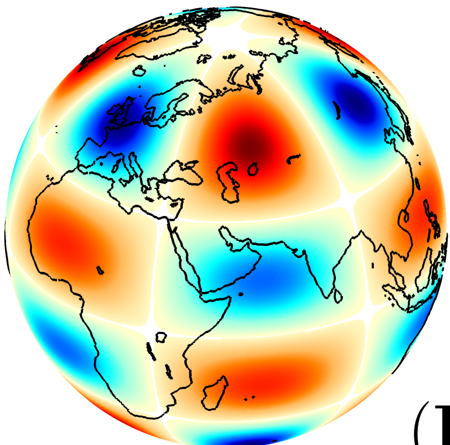


$(\mathbf{B}_{63})_\theta$

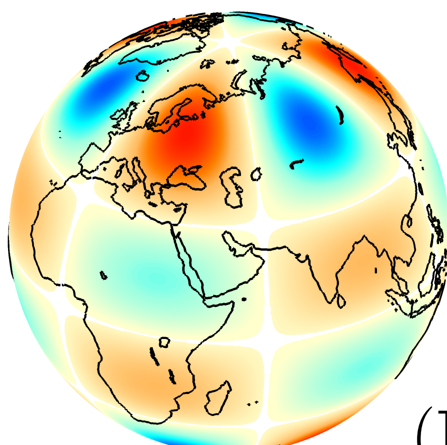


$(\mathbf{C}_{63})_\theta$

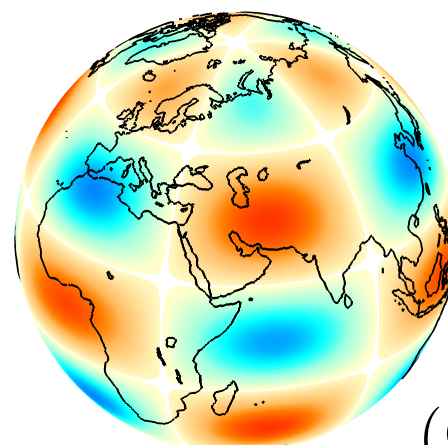
$$\mathbf{P}_{lm} = \hat{\mathbf{r}} Y_{lm}$$



$(\mathbf{P}_{63})_r$



$(\mathbf{B}_{63})_\phi$



$(\mathbf{C}_{63})_\phi$

New: ***Vectorial*** **Slepian functions — 1**

Let $\mathbf{g}$ be a **spectrally bandlimited *vector*** field:

and maximize the **spatial concentration** inside of the region of interest:

The **vector Shannon number**, or sum of the eigenvalues,

is the **effective dimension** space for which the vector $\mathbf{g}$ are a **basis.**

New: *Vectorial* Slepian functions — 1

Let $\mathbf{g}$ be a **spectrally bandlimited *vector*** field:

$$\mathbf{g} = \mathbf{g}^r + \mathbf{g}^t = \sum_{l=0}^{L} \sum_{m=-l}^{m} U_{lm}\mathbf{P}_{lm} + V_{lm}\mathbf{B}_{lm} + W_{lm}\mathbf{C}_{lm},$$

and maximize the **spatial concentration** inside of the region of interest:

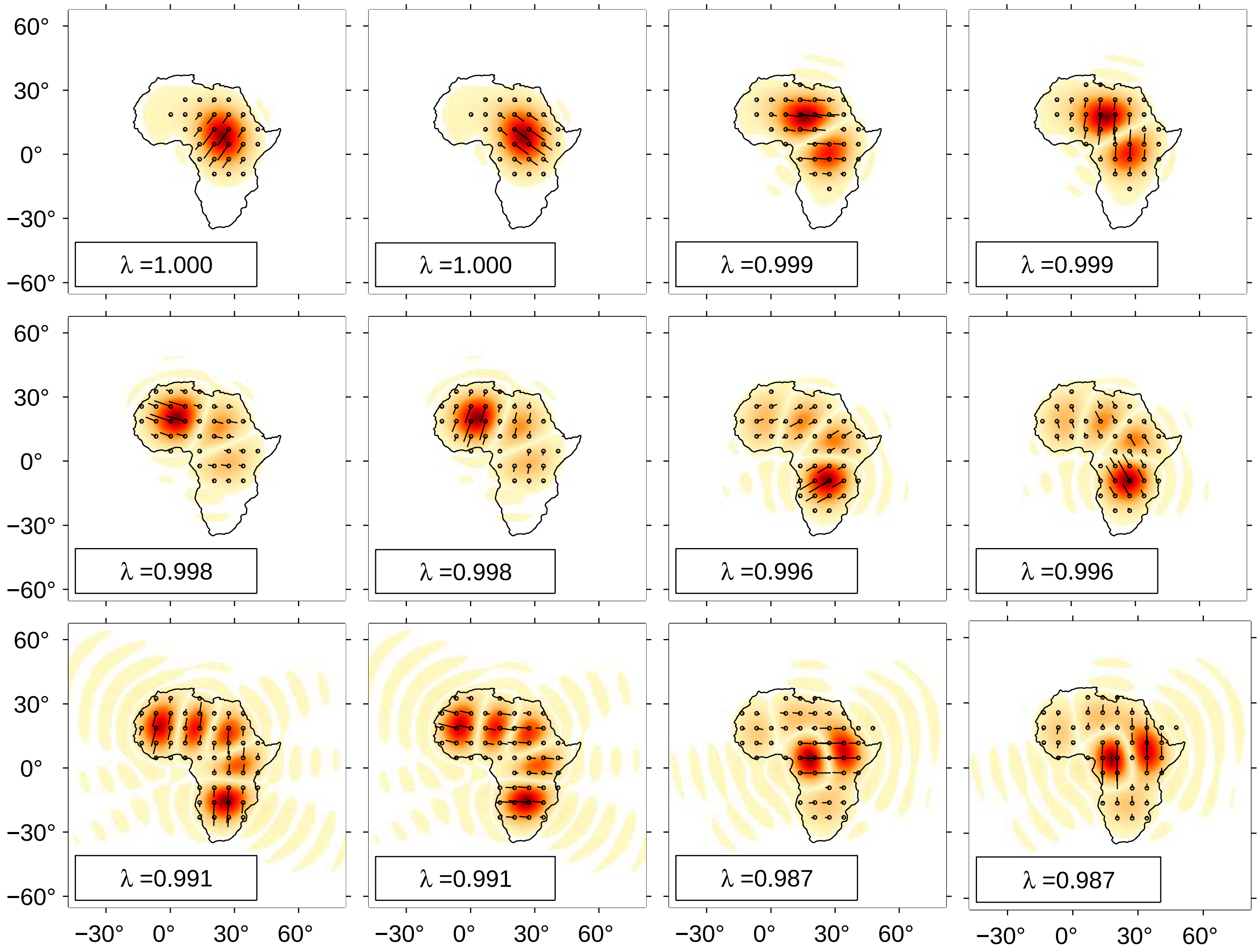
$$\lambda = \frac{\displaystyle\int_R \mathbf{g}\cdot\mathbf{g}\, d\Omega}{\displaystyle\int_\Omega \mathbf{g}\cdot\mathbf{g}\, d\Omega} = \text{maximum}.$$

The **vector Shannon number**, or sum of the eigenvalues,

$$K = [3(L+1)^2 - 2]\frac{A}{4\pi},$$

is the **effective dimension** of the space for which the vector $\mathbf{g}$ are a **basis**.

Vectorial Slepian functions — 2



Inverse problems in a *vector* Slepian basis — 1

Signal:

NGDC720 V3 (*Maus*, 2010)

$L = 72$

Data points:

2292 equal-area-random

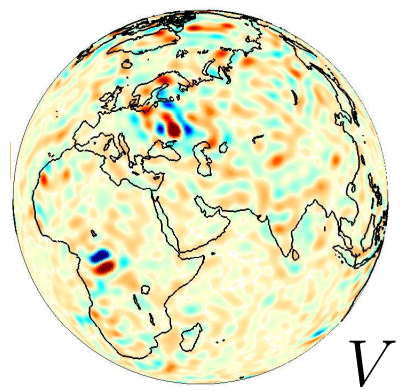
points over Africa

Noise:

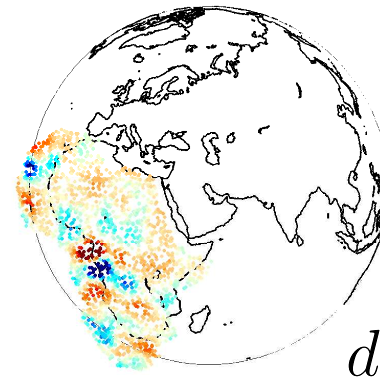
Gaussian random values

$\mu = 0$,

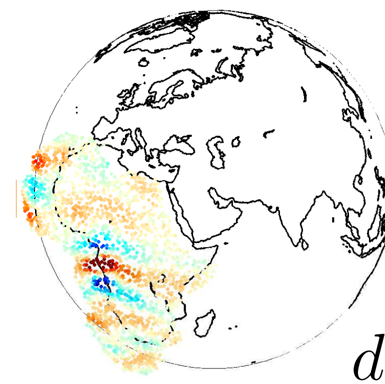
$\sigma^2 = 2.5\%$ of signal energy



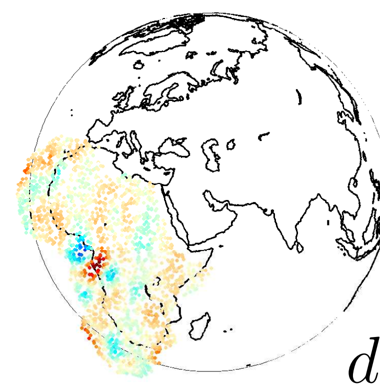
V



d_r



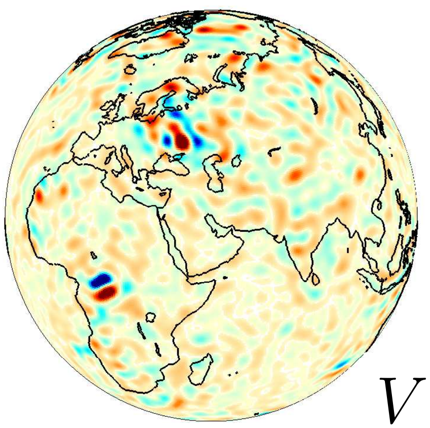
d_θ



d_ϕ

Inverse problems in a *vector* Slepian basis — 2

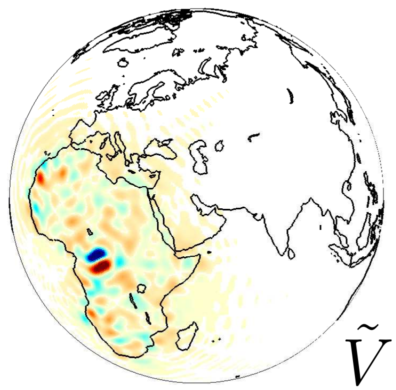
True potential



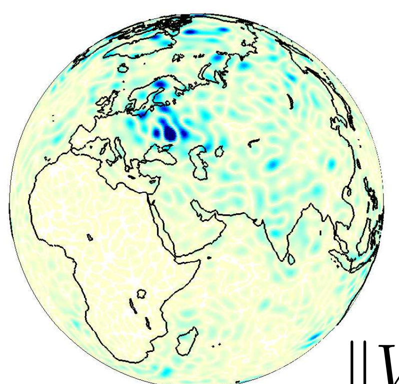
V

$a = 500$ km

$J = 386$



$\tilde{V}$

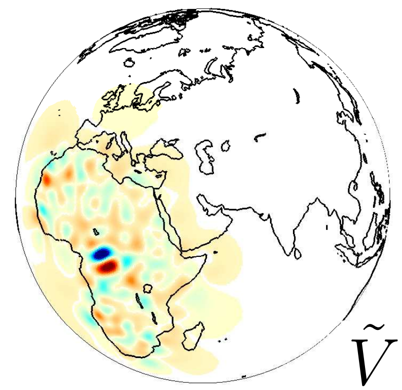


$\|V - \tilde{V}\|$

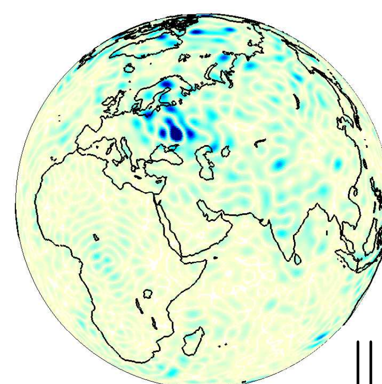
Relative mse=2.4%

$a = 800$ km

$J = 291$



$\tilde{V}$



$\|V - \tilde{V}\|$

Relative mse=18%