Spherical Signal Analysis

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In collaboration with Martin Büttner, Mike Hobson, Rod Kennedy, Zubair Khalid, Anthony Lasenby, Boris Leistedt, Daniel Mortlock, Hiranya Peiris, Gilles Puy, Jean-Philippe Thiran, Pierre Vandergheynst, Dimitri Van De Ville, & Yves Wiaux

Science on the Sphere
A Royal Society International Scientific Seminar, Chicheley Hall, July 2014

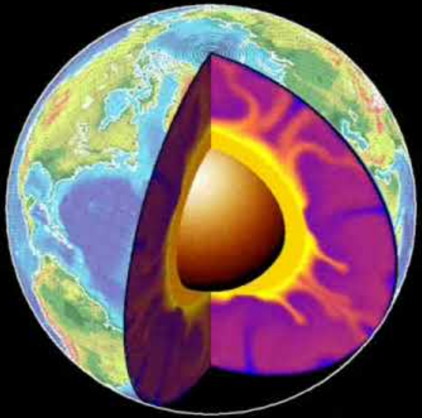
Observations on spherical manifolds

Earth

Credit: NASA

Observations on spherical manifolds

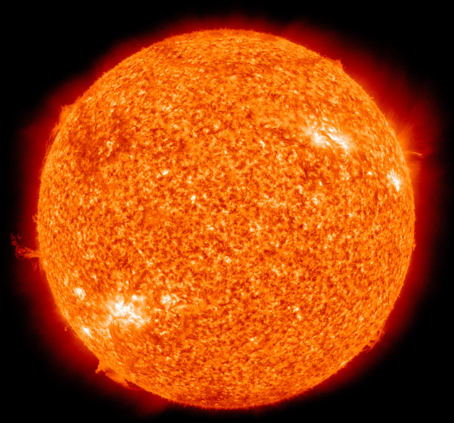
Earth's interior



Credit: http://maps.unomaha.edu/

Observations on spherical manifolds

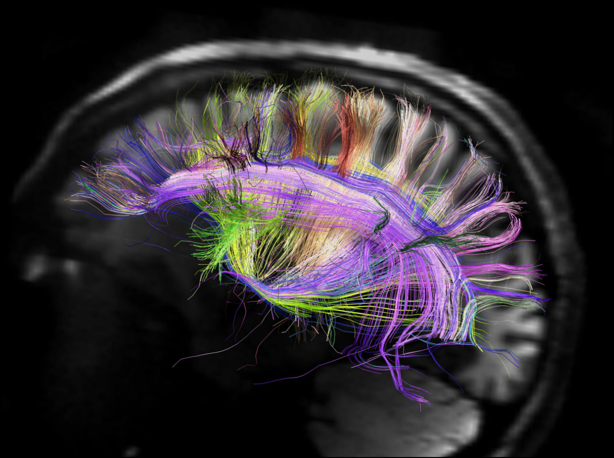
Sun



Credit: NASA

Observations on spherical manifolds

Diffusion magnetic resonance imaging



Credit: http://neuroimages.tumblr.com/

Observations on spherical manifolds

Computer graphics

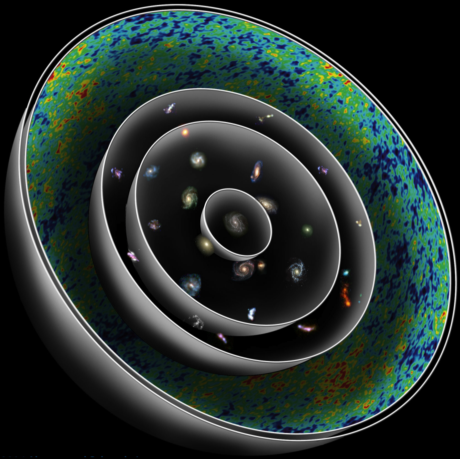
Credit: http://www.pauldebevec.com

Observations on spherical manifolds

Computer graphics

Observations on spherical manifolds

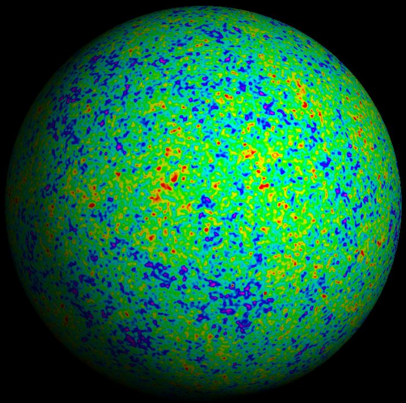
Cosmology



Credit: Abrams and Primack Inc.

Observations on spherical manifolds

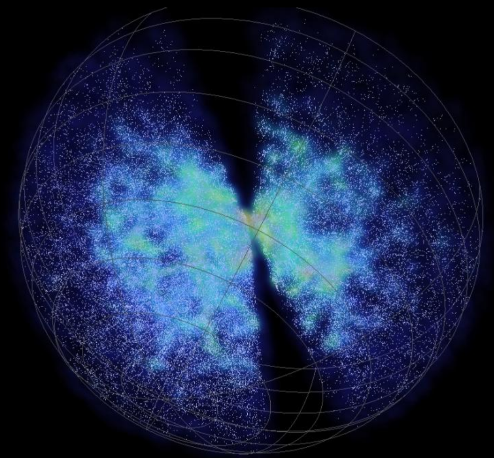
Cosmic microwave background (CMB) radiation



Credit: WMAP

Observations on spherical manifolds

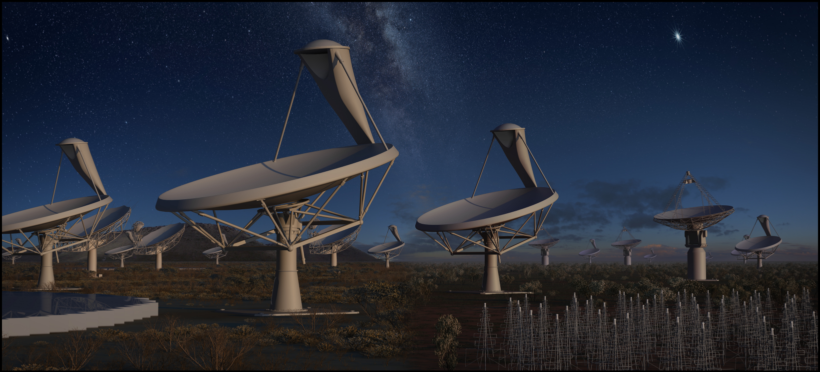
Galaxy surveys



Credit: SDSS

Observations on spherical manifolds

Radio interferometry



Credit: SKA Organisation

Outline

1. Sampling
2. Spatial-Spectral Concentration
3. Wavelets
4. Compressive Sensing
5. Optimal Filtering

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Sampling
Spherical harmonics

- Spherical harmonics are the eigenfunctions of the Laplacian on the sphere:

$$\Delta_{\mathbb{S}^2} Y_{\ell m} = -\ell(\ell+1) Y_{\ell m}$$

eigenfunctions

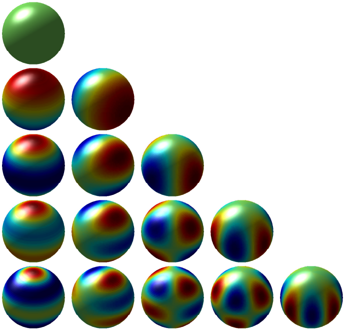

Figure: *Spherical harmonics* (real part) for $\ell, m \in \{0, 1, 2, 3\}$, $m \leq \ell$, with ℓ increasing down the rows from top to bottom and m increasing across the columns from left to right.

Sampling

Spherical harmonic transform and sampling theorems

- Function on the sphere $f \in \mathrm{L}^2(\mathbb{S}^2)$ may be represented by its spherical harmonic expansion:

$$f(\theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m} \, Y_{\ell m}(\theta, \varphi) \, ,$$

where the spherical harmonic coefficients are given by:

$$f_{\ell m} = \langle f, \, Y_{\ell m} \rangle = \int_{\mathbb{S}^2} \mathrm{d}\Omega(\theta, \varphi) \, f(\theta, \varphi) \, Y_{\ell m}^*(\theta, \varphi) \, .$$

- How do we sample a band-limited signal to capture all of its information content?

$\rightarrow$ Sampling theorems on the sphere

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Sampling

Practical sampling schemes on the sphere

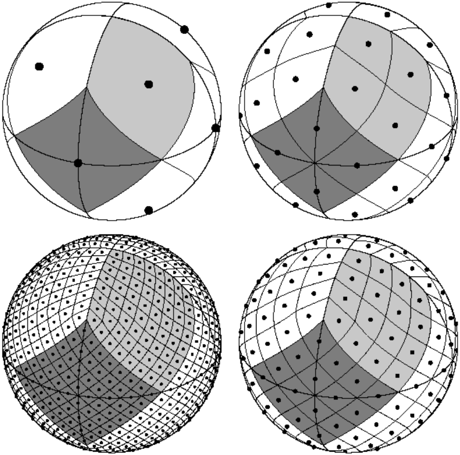

Figure: HEALPix pixelisation of the sphere (Gorski *et al.* 2005)

Sampling

Sampling theorems on the sphere

- From an information theoretic perspective, fundamental property is the number of samples required to capture the information content of signal band-limited at L.
- Optimal number of samples given by harmonic dimensionality of L^2.
- Equiangular sampling theorems:
  - Driscoll & Healy (1994): $4L^2$ samples
  - McEwen (2008): $2L^2$ samples ($+$spin but unstable)
  - Huffenberger & Wandelt (2010): $4L^2$ samples ($+$spin)
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- Optimal equiangular sampling scheme (but not sampling theorem):
  - Khalid, Kennedy & McEwen (2014): L^2 samples ($+$spin)

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Sampling

Sampling theorems on the ball

- Consider functions on the ball $\mathbb{B}^3 = \mathbb{R}^+ \times \mathbb{S}^2$, *i.e.* $f \in \mathrm{L}^2(\mathbb{B}^3)$.
- Fourier-Bessel functions are the canonical orthogonal basis on the ball since they are the eigenfunctions of the Laplacian:

$$X_{\ell m}(k, \boldsymbol{r}) = j_\ell(kr) Y_{\ell m}(\theta, \varphi).$$

Fourier-Bessel

- Fourier-Bessel transform of $f \in \mathrm{L}^2(\mathbb{B}^3)$ reads

$$\tilde{f}_{\ell m}(k) = \sqrt{\frac{2}{\pi}} \int_{\mathbb{B}^3} \mathrm{d}^3\boldsymbol{r}\, f(\boldsymbol{r})\, j_\ell^*(kr)\, Y_{\ell m}^*(\theta, \varphi),$$

where $\mathrm{d}^3\boldsymbol{r} = r^2 \sin\theta \,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi$ is the usual measure in spherical coordinates.

- Inverse transform given by

$$f(\boldsymbol{r}) = \sqrt{\frac{2}{\pi}} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_{\mathbb{R}^+} \mathrm{d}k k^2\, \tilde{f}_{\ell m}(k)\, j_\ell(kr)\, Y_{\ell m}(\theta, \varphi).$$

But does not admit a sampling theorem on the ball.

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But does not admit a sampling theorem on the ball.

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Sampling theorems on the ball

- Define the Fourier-Laguerre basis functions by

$$Z_{\ell mp}(\boldsymbol{r}) = K_p(r) Y_{\ell m}(\theta, \varphi),$$

Fourier-Laguerre

where radial basis defined by the spherical Laguerre functions $K_p(r) \propto e^{-r/2\tau} L_p^{(2)}(r/\tau)$ and $L_p^{(2)}$ is p-th generalised Laguerre polynomial of order two (Leistedt & McEwen 2012).

- A signal $f \in \mathrm{L}^2(\mathbb{B}^3)$ can then be decomposed as

$$f(\boldsymbol{r}) = \sum_{p=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell mp} \, Z_{\ell mp}(\boldsymbol{r}),$$

where the harmonic coefficients are given by the usual projection

$$f_{\ell mp} = \langle f | Z_{\ell mp} \rangle_{\mathbb{B}^3} = \int_{\mathbb{B}^3} \mathrm{d}^3 \boldsymbol{r} \, f(\boldsymbol{r}) \, Z^*_{\ell mp}(\boldsymbol{r}).$$

Affords exact and efficient harmonic transform on the ball.

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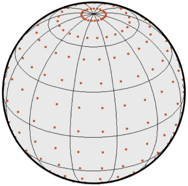
Sampling

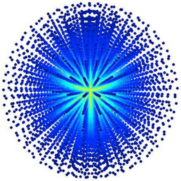
Codes



SSHT code: Spin spherical harmonic transforms

http://www.spinsht.org

A novel sampling theorem on the sphere
McEwen & Wiaux (2011)



FLAG code: Fourier-Laguerre transforms

http://www.flaglets.org

Exact wavelets on the ball
Leistedt & McEwen (2012)

Outline

Slepian spatial-spectral concentration

Formulation

- Spherical harmonics localised in spectral domain but have global spatial support.
- Spatial-spectral localisation trade-off.
- Given a region R, find the band-limited function f with energy concentrated in region R.
- Maximise the energy concentration:

$$\lambda = \frac{\int_R d^3\mu(\boldsymbol{r}) |f(\boldsymbol{r})|^2}{\int_{\mathbb{B}^3} d^3\mu(\boldsymbol{r}) |f(\boldsymbol{r})|^2}$$

concentration

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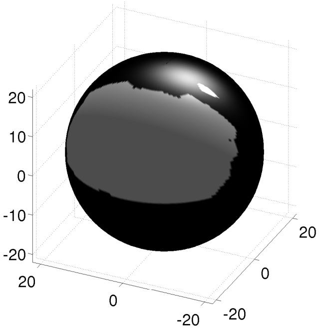

Figure: Non-trivial spatial region R (SDSS)

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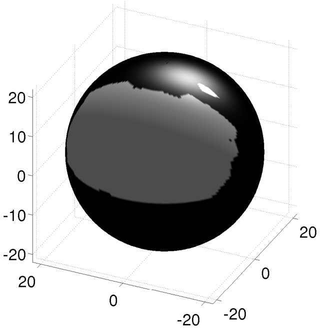

Figure: Non-trivial spatial region R (SDSS)

Slepian spatial-spectral concentration
Solution

- Solve eigenproblem to find band-limited, spatially concentrated functions:

$$S_R f = \lambda f$$

- Eigenvalue λ gives a measure of concentration.
- Dual problem: find space-limited, spectrally concentrated function.
- Spatial-spectral concentration on the sphere
  - Albertella, Sansò & Sneeuw (1999)
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  - Wieczorek & Simons (2005), Simons, Dahlen & Wieczorek (2006)
  - Khalid, Durrani, Kennedy & Sadeghi (2011)
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Slepian spatial-spectral concentration

Fourier-Bessel Slepian spatially concentrated functions

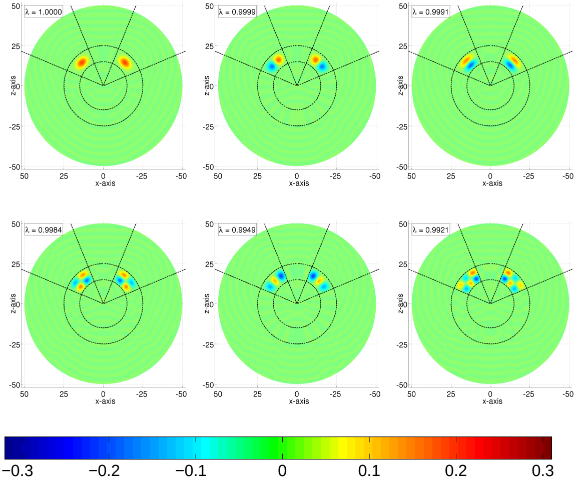


Figure: Fourier-Bessel band-limited spatially concentrated eigenfunctions.

Slepian spatial-spectral concentration

Fourier-Laguerre Slepian spatially concentrated functions

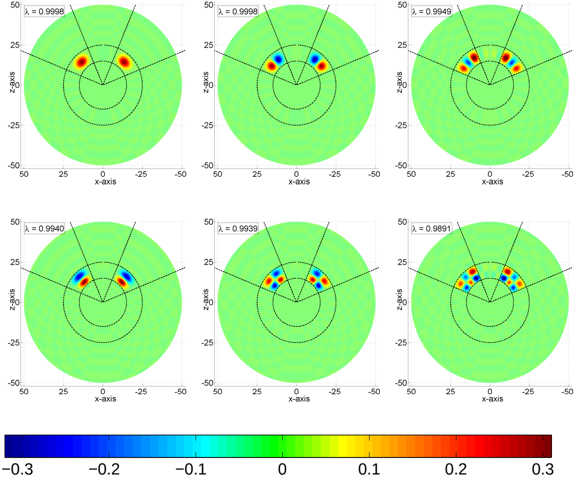


Figure: Fourier-Laguerre band-limited spatially concentrated eigenfunctions.

Slepian spatial-spectral concentration

Sparsity of Slepian decomposition

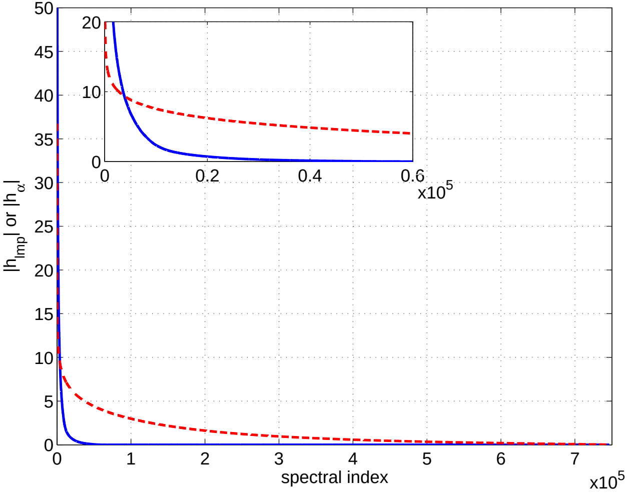
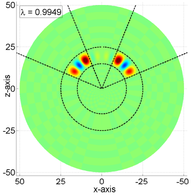


Figure: Spectral decay of the Fourier-Laguerre (red, dashed) and Slepian coefficients (blue, solid)

Slepian spatial-spectral concentration

Code



Slepian code: Slepian spatial-spectral concentration

Coming soon!

Slepian spatial-spectral concentration on the ball
Khalid, Kennedy & McEwen (2014)

Outline

1. Sampling
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3. Wavelets
4. Compressive Sensing
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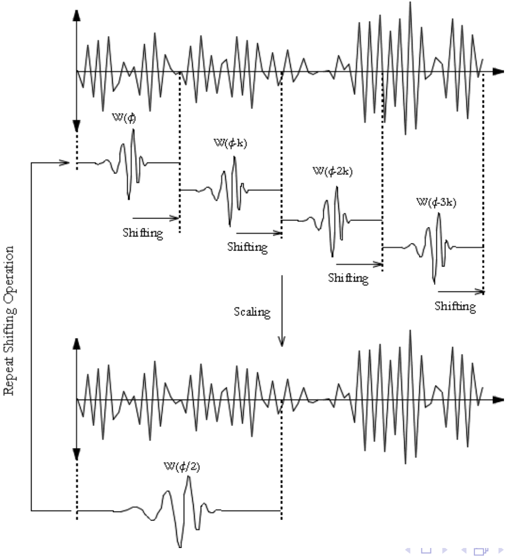
Wavelets

Recall wavelet transform in Euclidean space



Wavelets on the sphere

Dilation and translation

- Construct wavelet atoms from affine transformations (dilation, translation) on the sphere of a mother wavelet.
- The natural extension of translations to the sphere are rotations. Rotation of a function f on the sphere is defined by

$$[\mathcal{R}(\rho)f](\omega) = f(\mathsf{R}_\rho^{-1} \cdot \omega), \quad \omega = (\theta, \varphi) \in \mathbb{S}^2, \quad \rho = (\alpha, \beta, \gamma) \in \mathrm{SO}(3) \; .$$

translation

- How define dilation on the sphere?
  - Stereographic projection
    Antoine & Vandergheynst (1999), Wiaux *et al.* (2005)
  - Harmonic dilation wavelets
    McEwen *et al.* (2006), Sanz *et al.* (2006)
  - Isotropic undecimated wavelets
    Starck *et al.* (2005), Starck *et al.* (2009)
  - Needlets
    Narcowich *et al.* (2006), Baldi *et al.* (2009), Marinucci *et al.* (2008), Geller *et al.* (2008)
  - Scale-discretised wavelets
    Wiaux, McEwen, Vandergheynst, Blanc (2008)

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    Narcowich *et al.* (2006), Baldi *et al.* (2009), Marinucci *et al.* (2008), Geller *et al.* (2008)
  - Scale-discretised wavelets
    Wiaux, McEwen, Vandergheynst, Blanc (2008)

Wavelets on the sphere
Wavelet construction

- Fast algorithms:
  - McEwen, Hobson, Mortlock & Lasenby (2007), Wandelt & Gorski (2002), Risbo (1995)
  - Wiaux, Jacques, Vielva & Vandergheynst (2006)
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  - McEwen, Vandergheynst & Wiaux (2013)
- Scale-discretised wavelets: *Exact reconstruction with directional wavelets on the sphere* Wiaux, McEwen, Vandergheynst & Blanc (2008)
- Extend to spin functions (McEwen *et al.*, in prep.).

(a) Real$({}_2\Psi^j)$ (b) Imag$({}_2\Psi^j)$ (c) Abs$({}_2\Psi^j)$

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Wavelets on the sphere

Forward and inverse transform

- The spin scale-discretised wavelet transform is given by the usual projection onto each wavelet:

$$\underbrace{W^{{}_s\Psi^j}(\rho) = \langle {}_s f, \, \mathcal{R}_\rho \, {}_s\Psi^j \rangle}_{\text{projection}} = \int_{\mathbb{S}^2} \mathrm{d}\Omega(\omega) \, {}_s f(\omega) \, (\mathcal{R}_\rho \, {}_s\Psi^j)^*(\omega) \, .$$

- Wavelet coefficients are scalar and not spin.
- Wavelet coefficients live in $\mathrm{SO}(3) \times \mathbb{Z}$; thus, directional structure is naturally incorporated.

- The original function may be recovered exactly in practice from the wavelet (and scaling) coefficients:

$${}_s f(\omega) = \underbrace{\sum_{j=0}^{J}}_{\text{finite sum}} \underbrace{\int_{\mathrm{SO}(3)} \mathrm{d}\varrho(\rho) \, W^{{}_s\Psi^j}(\rho) (\mathcal{R}_\rho \, {}_s\Psi^j)(\omega) \, .}_{\text{wavelet contribution}}$$

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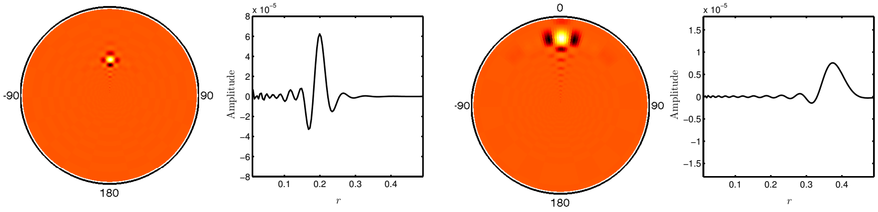
Wavelets on the ball
Fourier-LAGuerre wavelets (flaglets)

- *Exact wavelets on the ball*
  Leistedt & McEwen (2012)
- Angular (radial) aperture of localised functions is invariant under radial (angular) translation.
- Alternatives: Spherical 3D isotropic wavelets (Lanusse, Rassat & Starck 2012).



(a) Wavelet kernel translated by $r = 0.2$ (b) Wavelet kernel translated by $r = 0.4$

Figure: Slices of an axisymmetric flaglet wavelet kernel plotted on the ball of radius $R = 0.5$.

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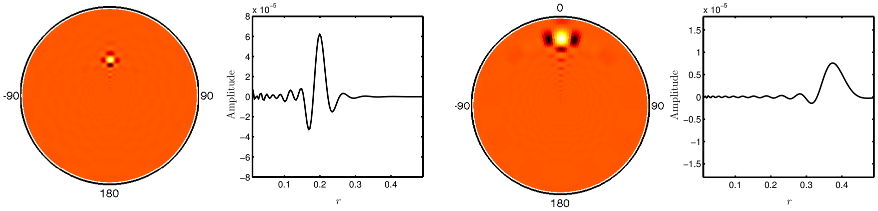

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Wavelets on the ball

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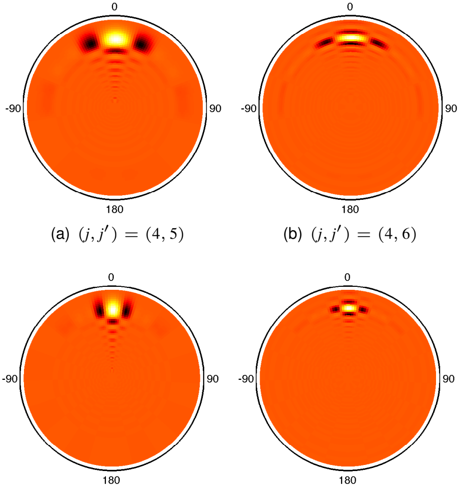

(a) $(j, j') = (4, 5)$ (b) $(j, j') = (4, 6)$

(c) $(j, j') = (5, 5)$ (d) $(j, j') = (5, 6)$

Figure: Scale-discretised wavelets on the ball.

Wavelets on the ball

Fourier-LAGuerre wavelets (flaglets)

- Fourier-Laguerre wavelet transform is given by the usual projection onto each wavelet:

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$$f(\boldsymbol{r}) = \underbrace{\sum_{j=J_0}^{J} \sum_{j'=J_0'}^{J'}}_{\text{finite sum}} \underbrace{\int_{B^3} \mathrm{d}^3\boldsymbol{r}' W^{\Psi^{jj'}}(\boldsymbol{r}') (\mathcal{T}_{\boldsymbol{r}} \Psi^{jj'})(\boldsymbol{r}') \,.}_{\text{wavelet contribution}}$$

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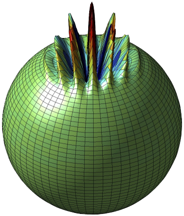
Wavelets on the sphere and ball

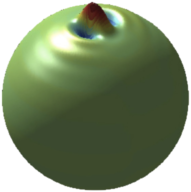
Codes



FastCSWT code

`http://www.fastcswt.org`

Fast directional continuous spherical wavelet transforms
McEwen, Hobson, Mortlock & Lasenby (2007)

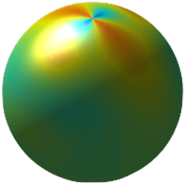


S2DW code

`http://www.s2dw.org`

Exact reconstruction with directional wavelets on the sphere
Wiaux, McEwen, Vandergheynst & Blanc (2008)
McEwen, Vandergheynst, & Wiaux (2013)

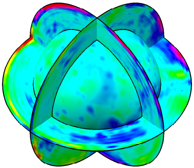
Wavelets on the sphere and ball
Codes



S2LET code

http://www.s2let.org

S2LET: A code to perform fast wavelet analysis on the sphere
Leistedt, McEwen, Vandergheynst, Wiaux (2012)



FLAGLET code

http://www.flaglets.org

Exact wavelets on the ball
Leistedt & McEwen (2012)

Outline

Compressive sensing and sparse reconstruction

Euclidean setting

- Ill-posed inverse problem:

$$\boldsymbol{y} = \Phi \boldsymbol{x} + \boldsymbol{n} = \Phi \Psi \boldsymbol{\alpha} + \boldsymbol{n} .$$

- Solve by imposing a regularising prior that the signal to be recovered is sparse in Ψ, *i.e.* solve the following ℓ_0 optimisation problem:

$$\boldsymbol{\alpha}^\star = \underset{\boldsymbol{\alpha}}{\arg\min} \|\alpha\|_0 \text{ such that } \|\boldsymbol{y} - \Phi\Psi\boldsymbol{\alpha}\|_2 \leq \epsilon ,$$

  where the signal is synthesising by $\boldsymbol{x}^\star = \Psi \boldsymbol{\alpha}^\star$.

- Solving this problem is difficult (combinatorial).
- Instead, solve the ℓ_1 optimisation problem (convex):

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Compressive sensing and sparse reconstruction

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- Compressive sensing on the sphere:
  - Rauhut & Ward (2011)
  - Burq, Dyatlov, Ward & Zworski (2012)
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Outline

Optimal filters
Formulation

- Observed field model:

$$y(\omega) = \sum_i s_i(\omega) + n(\omega) \ ,$$

model

where each source is represented by its amplitude A_i and profile, $s_i(\omega) = A_i \, \tau_i(\omega)$, and $\tau_i(\omega)$ is a dilated and rotated version of the source profile $\tau(\omega)$ of default dilation centred on the north pole, *i.e.* $\tau_i(\omega) = \mathcal{R}(\rho_i) \, \mathcal{D}(R_i|p) \, \tau(\omega)$.

- Recover parameters of each source $\{A_i, R_i, \rho_i\}$ that describe amplitude, scale and position/orientation respectively.

- Filter the signal on the sphere to enhance the source profile relative to the background:

$$w(\rho, R) = \int_{\mathbb{S}^2} \mathrm{d}\Omega(\omega) \, f(\omega) \, [\mathcal{R}(\rho)\varphi_R]^*(\omega) \ ,$$

filtering

where $\varphi \in L^2(\mathbb{S}^2)$ is the filter kernel.

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Solution

- Matched filters applied extensively in Euclidean space (*e.g.* the plane) to enhance a source profile in a background noise process (*e.g.* Sanz *et al.* 2001, Herranz *et al.* 2002).
- Extend optimal filtering to the sphere:
  - Tegmark & de Oliveira-Costa (1998): point sources
  - Schafer, Pfrommer, Hell & Bartelmann (2006): axisymmetric
  - McEwen, Hobson & Lasenby (2008): directional

Matched filter (MF) on the sphere

The optimal MF defined on the sphere is obtained by solving the constrained optimisation problem:

$$\min_{\varphi_R} \sigma_w^2(\mathbf{0}, R) \quad \text{such that} \quad \langle w(\mathbf{0}, R) \rangle = A \,.$$

The spherical harmonic coefficients of the resultant MF are given by

$$(\varphi_R)_{\ell m} = \frac{\tau_{\ell m}}{a\, C_\ell} \,, \quad \text{where } a = \sum_{\ell m} C_\ell^{-1} |\tau_{\ell m}|^2 \,.$$

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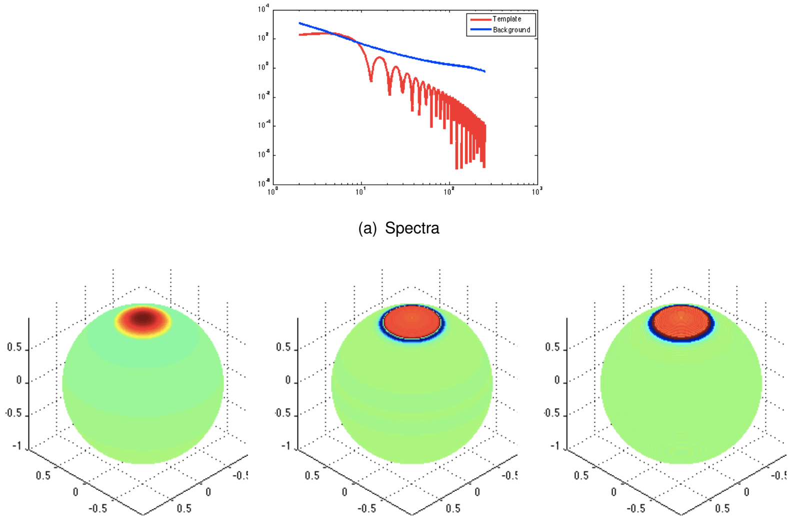
Optimal filters

Example

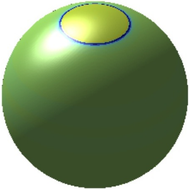


(a) Spectra

(b) Template (c) MF (d) SAF

Figure: Optimal filters for bubble template with size $\theta_{\rm crit} = 20^\circ$.

Optimal filters
Code



S2FIL code: Optimal filtering on the sphere

`http://www.jasonmcewen.org/codes/s2fil/doc/index_s2fil.html`

Optimal filters on the sphere
McEwen, Hobson & Lasenby (2008)

Summary

Spherical signal processing and analysis is beginning to become a mature field, with widespread application.

- Sampling
- Spatial-spectral concentration
- Wavelets
- Compressive sensing
- Optimal filtering
- Others...

- Application in cosmology and beyond...