

Kernel-Based Formulations of Spatio-Spectral Transform and Three Related Transforms on the Sphere

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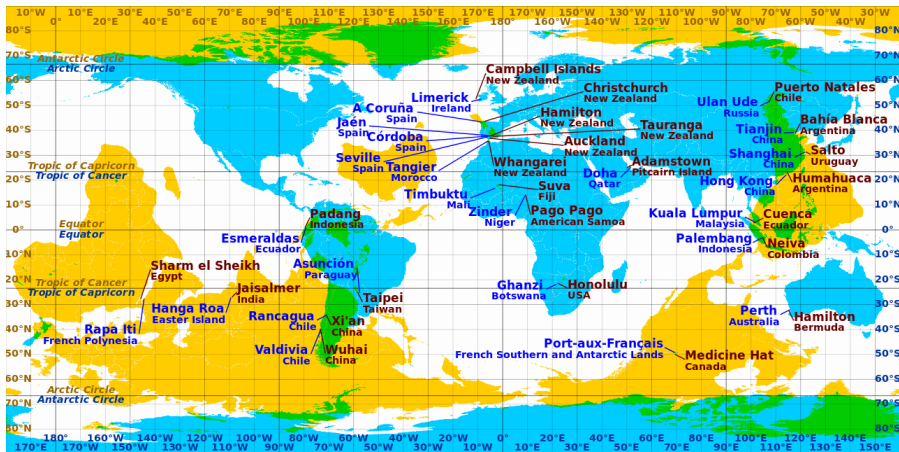
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Past work in spherical harmonics

- Radial transformation of RF feed horn patterns — radio astronomy cassegrain reflector systems (ATNF)
- Broadband and near-field arrays and beamforming — audio/acoustic applications
- Head-related transfer functions (HRTF) — hearing
- 3D-audio surround sound encoding and reproduction — multi eigen-channel rather than multi-channel
- 3D-audio microphone capture — spherical microphones
- MIMO (multi-antenna) communication channel modelling — 3D modal multipath modeling

Led to interest in **generic spherical methods** with many applications and **trawling SP and IT methods** for spherical signal applications.

- Slepian 1960s — Slepian on Sphere 1990's
- Time-frequency Methods — linear and Wigner-Ville (Cohen)
- Signals and Systems — functions and operators

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spatial in $L^2(\mathbb{S}^2) \longrightarrow$ spectral in ℓ^2

For a complex-valued spatial signal on the 2-sphere, $f(\hat{\mathbf{x}})$, the **Spherical Harmonic Transform (SHT)** is given by

$$(f)_\ell^m := \langle f, Y_\ell^m \rangle = \int_{\mathbb{S}^2} f(\hat{\mathbf{y}}) \overline{Y_\ell^m(\hat{\mathbf{y}})} ds(\hat{\mathbf{y}}).$$

This is the natural Fourier transform on the sphere.

spatial in $L^2(\mathbb{S}^2)$ \longleftarrow spectral in ℓ^2

The **Inverse Spherical Harmonic Transform (ISHT)** is given by

$$f(\hat{\mathbf{x}}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (f)_{\ell}^m Y_{\ell}^m(\hat{\mathbf{x}}),$$

where $(f)_{\ell}^m := \langle f, Y_{\ell}^m \rangle$. This is the inverse Fourier transform on the sphere.

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spatial \longrightarrow (joint) spatio-spectral on $\mathbb{S}^2 \times \mathbb{Z}$

$$f(\hat{\mathbf{x}}) \longrightarrow g(\hat{\mathbf{x}}; \ell, m), \quad \text{using symmetric window } h(\hat{\mathbf{x}})$$

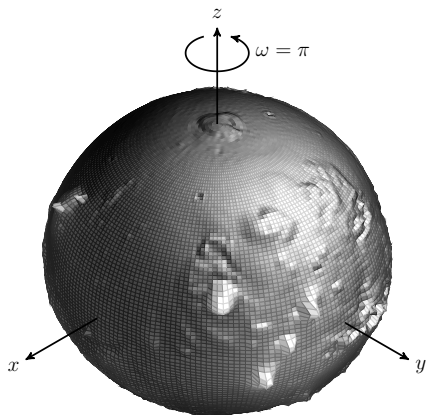
The **spatio-spectral SLSHT** [1, 2] was originally given by

$$g(\hat{\mathbf{x}}; \ell, m) := \int_{\mathbb{S}^2} (\mathcal{D}(\hat{\mathbf{x}})h)(\hat{\mathbf{y}}) f(\hat{\mathbf{y}}) \overline{Y_\ell^m(\hat{\mathbf{y}})} ds(\hat{\mathbf{y}}), \quad (1)$$

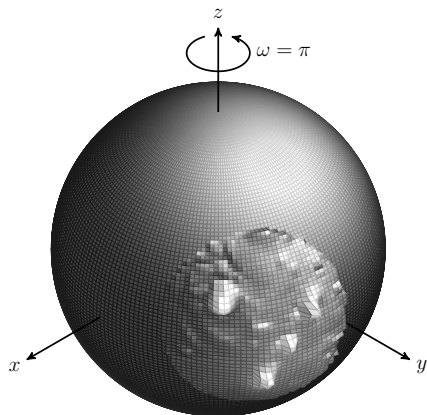
where $h(\hat{\mathbf{y}})$ is an azimuthally symmetric window function satisfying

$$\langle h, Y_\ell^m \rangle = 0, \quad \forall m \neq 0. \quad (2)$$

$(\mathcal{D}(\hat{\mathbf{x}})h)(\hat{\mathbf{y}})$ is the window $h(\hat{\mathbf{y}})$ rotated/centered about the point $\hat{\mathbf{x}} \in \mathbb{S}^2$.

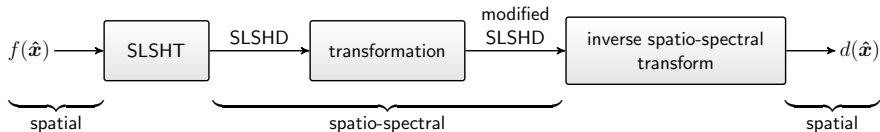


$$f(\hat{\mathbf{y}})$$



$$(\mathcal{D}(\hat{\mathbf{x}})h)(\hat{\mathbf{y}})f(\hat{\mathbf{y}})$$

Why?



$$\int_{\mathbb{S}^2} f(\hat{\mathbf{y}}) \overline{Y_\ell^m(\hat{\mathbf{y}})} ds(\hat{\mathbf{y}}) \quad \text{SHT of } f$$

$$\int_{\mathbb{S}^2} h(\hat{\mathbf{y}}) f(\hat{\mathbf{y}}) \overline{Y_\ell^m(\hat{\mathbf{y}})} ds(\hat{\mathbf{y}}) \quad \text{SHT of } f \text{ with window}$$

$$\int_{\mathbb{S}^2} (\mathcal{D}(\hat{\mathbf{x}})h)(\hat{\mathbf{y}}) f(\hat{\mathbf{y}}) \overline{Y_\ell^m(\hat{\mathbf{y}})} ds(\hat{\mathbf{y}}) \quad \text{SHT of } f \text{ with rotated window}$$

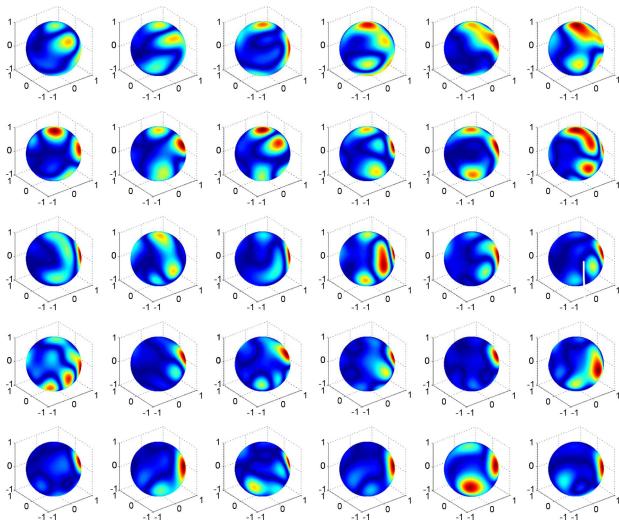
$$\int_{\mathbb{S}^2} (\mathcal{D}(\hat{\mathbf{x}})h)(\hat{\mathbf{y}}) f(\hat{\mathbf{y}}) \overline{Y_\ell^m(\hat{\mathbf{y}})} ds(\hat{\mathbf{y}}) \quad \text{SLSHT spatio-spectral arguments}$$

Evaluate this over all rotations $\hat{\mathbf{x}}$ and all spectral indices ℓ, m and this gives you a spatio-spectral domain representation of f

- parameterised by the window h
- analog of short-time Fourier Transform
- can generalise to directional windows and SO(3) rotations with “fast” computational methods [3]
- information preserving with inverse (to recover f)
- spatio-spectral domain processing generally needs pseudo-inverse [4]

$$\mathbb{S}^2 \times \mathbb{Z}$$

Insanity, really, . . .



$$\int_{\mathbb{S}^2} (\mathcal{D}(\hat{\mathbf{x}})h)(\hat{\mathbf{y}}) f(\hat{\mathbf{y}}) \overline{Y_\ell^m(\hat{\mathbf{y}})} ds(\hat{\mathbf{y}})$$

is not the best notation for the SLSHT because it really means two equations

$$g(\hat{\mathbf{x}}; \ell, m) := \int_{\mathbb{S}^2} (\mathcal{D}(\hat{\mathbf{x}})h)(\hat{\mathbf{y}}) f(\hat{\mathbf{y}}) \overline{Y_\ell^m(\hat{\mathbf{y}})} ds(\hat{\mathbf{y}}),$$

where $h(\hat{\mathbf{y}})$ is an azimuthally symmetric window function satisfying

$$\langle h, Y_\ell^m \rangle = 0, \quad \forall m \neq 0.$$

- This **symmetry isn't well-exploited** in the expression for the SLSHT.
- **Two** equations.
- SLSHT processing when using this formulation [4] **isn't intuitive**.

The **spatio-spectral SLSHT** $g(\hat{\mathbf{x}}; \ell, m)$ has various interpretations. The window $h(\hat{\mathbf{x}})$ is chosen to concentrate analysis into a local region on the sphere centered on the point $\hat{\mathbf{x}}$:

- $g_{\hat{\mathbf{x}}_0}(\ell, m) := g(\hat{\mathbf{x}}_0; \ell, m)$ — fixing the spatial $\hat{\mathbf{x}}_0 \in \mathbb{S}^2$ and varying the spectral degree ℓ and order m we get information of **which spherical harmonics contribute most to explain that localized portion of the spatial signal $f(\hat{\mathbf{x}})$ within the windowed region**; and
- $g_{\ell_0, m_0}(\hat{\mathbf{x}}) := g(\hat{\mathbf{x}}; \ell_0, m_0)$ — fixing both the spectral degree ℓ_0 and order m_0 and varying spatially $\hat{\mathbf{x}} \in \mathbb{S}^2$ we can infer from **which parts of the sphere the signal most strongly contribute to the (global) spherical harmonic coefficient**.
- tears apart the spatial signal on the sphere \mathbb{S}^2 into a cartesian product domain $\mathbb{S}^2 \times \mathbb{Z}$
- but not the only way to move the signal to a cartesian domain

- Seems slightly different to short-time Fourier Transform where the cartesian product domain is $\mathbb{R} \times \mathbb{R}$ for time-frequency. Somewhat symmetric, or self-dual or something.
- For spatio-spectral SLSHT the two domains \mathbb{S}^2 and \mathbb{Z} seem world's apart. (Here, recall, \mathbb{Z} is the single countable indexing of degrees ℓ and orders m .)
- Let's fix that.

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The **Spatio-Spectral SLSHT** [1] given by (1) and (2) can be written

$$g(\hat{\mathbf{x}}; \ell, m) = \int_{\mathbb{S}^2} H(\hat{\mathbf{x}} \cdot \hat{\mathbf{y}}) f(\hat{\mathbf{y}}) \overline{Y_\ell^m(\hat{\mathbf{y}})} ds(\hat{\mathbf{y}}), \quad (3)$$

where the **kernel** is

$$H(\hat{\mathbf{x}} \cdot \hat{\mathbf{y}}) := \sum_{\ell=0}^{\infty} (h)_\ell^0 \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\hat{\mathbf{x}} \cdot \hat{\mathbf{y}}) \quad (4)$$

and coefficients $(h)_\ell^0 = \langle h, Y_\ell^0 \rangle$.

- $H(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = H(\hat{\mathbf{x}} \cdot \hat{\mathbf{y}})$ is an isotropic convolution kernel.
- Follows from the Spherical Harmonic Addition Theorem.
- (3) is self-contained and completely defines the Spatio-Spectral SLSHT.
- (4) just shows the relationship to the symmetric spatial window $h(\hat{\mathbf{x}})$.
- Not a profound improvement but cleaner and simpler.

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Now for something weird. . .

- Fourier Transform, Spherical Harmonic Transform, Spatio-Spectral SLSHT are **Integral Transforms**
- (Also for discrete spectra the inverse transforms being series are essentially Integral Transform-like)
- So it is strange to get, next, a non-integral transform. In fact it looks odd, simple and probably useless (or is it?).
- (It has an odd-looking name.)

The **Spatio-Spatial SLSHT** is defined by

$$g(\hat{\mathbf{x}}; \hat{\mathbf{z}}) := \sum_{\ell, m} g(\hat{\mathbf{x}}; \ell, m) Y_{\ell}^m(\hat{\mathbf{z}}), \quad \hat{\mathbf{x}}, \hat{\mathbf{z}} \in \mathbb{S}^2, \quad (5)$$

which is the ISHT in the **2nd argument** of (3).

In kernel form, **Kernel-Based Spatio-Spatial SLSHT**,

$$g(\hat{\mathbf{x}}; \hat{\mathbf{z}}) := H(\hat{\mathbf{x}} \cdot \hat{\mathbf{z}}) f(\hat{\mathbf{z}}), \quad \hat{\mathbf{x}}, \hat{\mathbf{z}} \in \mathbb{S}^2, \quad (6)$$

- No integral. No Integral Transform.
- Just the **integrand** in the Kernel-Based Spatio-Spectral SLSHT (3)
- Here the cartesian product domain is $\mathbb{S}^2 \times \mathbb{S}^2$ — didn't we want that?
- Holds **the same information** as the original Spatio-Spectral SLSHT

$$g(\hat{\mathbf{x}}; \hat{\mathbf{z}}) := H(\hat{\mathbf{x}} \cdot \hat{\mathbf{z}})f(\hat{\mathbf{z}}), \quad \hat{\mathbf{x}}, \hat{\mathbf{z}} \in \mathbb{S}^2,$$

- $\hat{\mathbf{x}}$ is the spatial portion ("Spatio-")
- $\hat{\mathbf{z}}$ is the **spatial representation of the spectral portion** ("Spatial")
- on $\mathbb{S}^2 \times \mathbb{S}^2$
- inversion example:

$$f(\hat{\mathbf{x}}) = \frac{1}{H(1)}g(\hat{\mathbf{x}}; \hat{\mathbf{x}}), \quad \text{where } H(1) \neq 0$$

- leads to fast new spatio-spectral transforms and inverses [5]
- leads to $O(L)$ improvement over existing (non-symmetric) SS-algorithms

Another transform.

The **Kernel-Based Spectro-Spatial SLSHT** of a signal $f(\cdot)$ can be written

$$g(p, q; \hat{\mathbf{z}}) := f(\hat{\mathbf{z}}) \int_{\mathbb{S}^2} H(\hat{\mathbf{x}} \cdot \hat{\mathbf{z}}) \overline{Y_p^q(\hat{\mathbf{x}})} ds(\hat{\mathbf{x}}), \quad \hat{\mathbf{z}} \in \mathbb{S}^2, \quad (7)$$

using the isotropic convolution kernel $H(\cdot)$, given in (4).

- p, q is the **spectral representation of the spatial portion** ("Spectro-")
- $\hat{\mathbf{z}}$ is the **spatial representation of the spectral portion** ("Spatial")
- LSD-version

Yet another transform.

The **Kernel-Based Spectro-Spectral SLSHT** of a signal $f(\cdot)$ can be written

$$g(p, q; \ell, m) := \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} H(\hat{\mathbf{x}} \cdot \hat{\mathbf{y}}) f(\hat{\mathbf{y}}) \overline{Y_\ell^m(\hat{\mathbf{y}})} \overline{Y_p^q(\hat{\mathbf{x}})} ds(\hat{\mathbf{y}}) ds(\hat{\mathbf{x}}), \quad (8)$$

using the isotropic convolution kernel $H(\cdot)$, given in (4).

- p, q is the **spectral representation of the spatial portion** ("Spectro-")
- ℓ, m is the spectral portion ("Spectral")
- cartesian product domain is $\mathbb{Z} \times \mathbb{Z}$ (nice)
- naturally get here computationally but tends to be a false-friend

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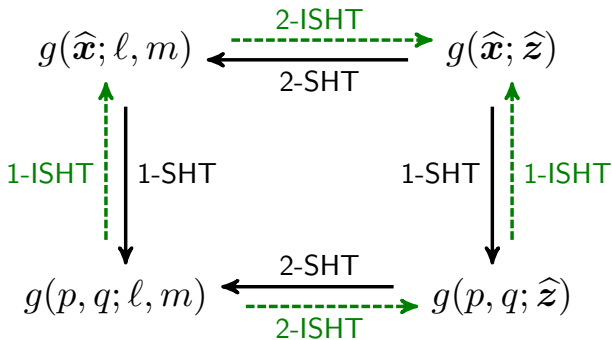


Figure: Transformations between the four variants of the SLSHT. In notation, 1-(I)SHT/2-(I)SHT denotes the (I)SHT on the 1st/2nd argument.

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- Write the integral transform using kernel K as \mathcal{L}_K . With K positive definite then we can work in a “smoother and smaller” space






$$\mathcal{L}_K^{1/2} : f \in L^2(\mathbb{S}^2) \longrightarrow h = \mathcal{L}_K^{1/2} f$$

This describes all the functions in the new space as the **low-pass filtered** versions of finite energy functions.

- If the filtering is strong enough (Hilbert-Schmidt) then we can manufacture RKHS's.
- Strict band-limiting to degree L is a degenerate version of this, ends up being a subspace.
- The square root operator generates of new smooth space that generally is **not** a subspace of $L^2(\mathbb{S}^2)$. The inner-product is different to incorporate the decay of the eigenvalues.

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- Can generalise the concentration measures in spatial and spectral domains, without damaging the nice properties — incorporate different tapers.
- “Remarkable” dual-orthogonality of Slepian eigen-functions is a general feature of an abstract problem, not a happy accident. So there are Slepian-like alternatives, for example, which might suppress the wildness of the eigenfunctions near the edges of spatial regions.

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